Exercises by Maxima/wxMaxima. The following exercises are accompanied by Maxima code files which guide you to the correct answers.

1. Prove the curvature formula of Theorem 10 in Section 13.3 (`e-math/2110/theorem13-3.wxm`)
   (a) Prove the following derivative formula for the unit tangent vector $\mathbf{T}$.
   $$|\mathbf{T}'| = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^2}$$
   (b) Prove Theorem 10 in Section 13.3.
2. Differentiation with vector operations (`e-math/2110/exercise13-3.wxm`)
   (a) Reparametrize the curve $\langle e^{2t}\cos 2t, 2, e^{2t}\sin 2t\rangle$ to arc length.
   (b) Find the unit tangent and unit normal vectors, and find the curvature for the curve $\langle t^2, \sin t - t\cos t, \cos t + t\sin t\rangle$ (Section 13.3: Exercise 19).
   (c) Find the curvature for the curve $\langle t, t, 1+t^2\rangle$.
   (d) Find the curvature for $y = \cos x$.
   (e) Find the vectors $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ for $\langle \cos t, \sin t, \ln\cos t\rangle$ at $t = 0$ (Section 13.3: Exercise 52).
   (f) Find the point on the curve $\langle t^3, 3t, t^4\rangle$ where the normal plane is parallel to $6x+6y-8z = 1$ (Section 13.3: Exercise 57).
3. Differentiation/integration with vector operations (`e-math/2110/exercise13-4.wxm`)
   (a) Find the velocity, acceleration and speed of the position $\langle 2\cos t, 3t, 2\sin t\rangle$ (Section 13.4: Exercise 10).
   (b) Find the velocity and position given the acceleration $\langle 2, 6t, 12t^2\rangle$ and given the initial velocity $\langle 1, 0, 0\rangle$ and the initial position $\langle 0, 1, -1\rangle$.
   (c) Find the velocity and position given the acceleration $\langle t, e^t, e^{-t}\rangle$ and given the initial velocity $\langle 0, 0, 1\rangle$ and the initial position $\langle 0, 1, 1\rangle$. Then draw the path of positions (Section 13.4: Exercise 18).
   (d) Find the tangential and normal components of acceleration for the position $\langle 1+t, t^2-2t, 0\rangle$. And draw the path of positions.

Key formulas to memorize:

1. $s(t) = \displaystyle\int_0^t |\mathbf{r}'(u)|\, du$ (the arc length of the space curve $\mathbf{r}(t)$ from 0 to t)
2. $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ (the unit tangent vector);

   $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$ (the curvature)

3. $|\mathbf{T}'(t)| = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^2}$ (the derivative of unit tangent vector);

   $\kappa(t) = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$ (the curvature formula)

4. $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$ (normal vector);

   $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ (binormal vector)

5. $\mathbf{r}''(t) = a_T\mathbf{T}(t) + a_N\mathbf{N}(t)$;

   $a_T = \dfrac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|}$ and $a_N = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|}$ (tangential and normal component)

Online quiz No.3: Complete online quiz at the course website.

Exercises discussed in class. Read "Kepler's laws of planetary motion" in Section 13.4, and discuss the Kepler's first law: The acceleration $\mathbf{r}''(t)$ of a planet is determined by

$$\mathbf{r}''(t) = -\frac{GM}{r^3(t)}\mathbf{r}(t) = -\frac{GM}{r^2(t)}\mathbf{u}(t) \tag{1}$$

and the position $\mathbf{r}(t)$ revolves around the sun at the origin $(0,0,0)$ in an elliptical orbit. Here $r(t)$ denotes the distance $|\mathbf{r}(t)|$ between the sun and the planet, and

$$\mathbf{u}(t) = \frac{\mathbf{r}(t)}{r(t)} \tag{2}$$

denotes the unit vector in the direction of $\mathbf{r}(t)$ to the planet. We prove this Newton's assertion by completing the following questions.

1. Assuming $0 < e < 1$, show that the polar equation

   $$r = \frac{a(1-e^2)}{1+e\cos\theta} \tag{3}$$

   represents an ellipse having foci $(0,0)$ and $(-2ae,0)$ with vertices $(a(1-e),0)$ and $(-a(1+e),0)$ (Section 10.6). That is, show that $(x,y) = (r\cos\theta, r\sin\theta)$ satisfies

   $$\frac{(x+ae)^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1.$$

2. By applying (1) show that

   $$\frac{d}{dt}[\mathbf{r}(t) \times \mathbf{r}'(t)] = \mathbf{0}.$$

   Then argue that $\mathbf{r}(t) \times \mathbf{r}'(t)$ is a constant vector. This is the law of conservation of angular momentum; here $\mathbf{r}(t) \times m\mathbf{r}'(t)$ is called "angular momentum" (see Exercise 44 of Section 13.4).

3. Let $\mathbf{h}$ be the constant vector so that $\mathbf{h} = \mathbf{r}(t) \times \mathbf{r}'(t)$. By using (2) show that

$$\mathbf{h} = \mathbf{r}(t) \times \mathbf{r}'(t) = r^2(t)\mathbf{u}(t) \times \mathbf{u}'(t) \tag{4}$$

Let $h = |\mathbf{h}|$. Here we can observe that $r^2(t)|\mathbf{u}'(t)|/2 = h/2$; the Kepler's second law (see "Applied Project" of Section 13.4).

4. By applying (1) show that

$$\mathbf{r}''(t) \times \mathbf{h} = GM\mathbf{u}'(t) \tag{5}$$

Hint: Show that $\mathbf{u}(t) \times [\mathbf{u}(t) \times \mathbf{u}'(t)] = -\mathbf{u}'(t)$.

5. Argue that by (5) there exists a constant vector $\mathbf{c}$ such that $\mathbf{c}$ is perpendicular to $\mathbf{h}$, and it satisfies

$$\mathbf{r}'(t) \times \mathbf{h} = GM\mathbf{u}(t) + \mathbf{c} \tag{6}$$

6. Let $c = |\mathbf{c}|$ be the length of $\mathbf{c}$, and let

$$\mathbf{i} = \frac{\mathbf{c}}{c} \quad \text{and} \quad \mathbf{j} = \frac{\mathbf{h} \times \mathbf{c}}{hc}$$

be unit orthogonal vectors perpendicular to $\mathbf{h}$. Set the angle $\theta(t)$ between $\mathbf{i}$ and $\mathbf{r}(t)$ so that

$$\mathbf{r}(t) = (r(t)\cos\theta(t))\mathbf{i} + (r(t)\cos\theta(t))\mathbf{j}. \tag{7}$$

Show that by (6) we obtain

$$\mathbf{r}(t) \cdot [\mathbf{r}'(t) \times \mathbf{h}] = GMr(t) + cr(t)\cos\theta(t)$$

7. Use the triple product formula to $\mathbf{r}(t) \cdot [\mathbf{r}'(t) \times \mathbf{h}]$, and apply (4). Then show that

$$\mathbf{r}(t) \cdot [\mathbf{r}'(t) \times \mathbf{h}] = h^2$$

and therefore, that

$$h^2 = GMr(t) + cr(t)\cos\theta(t) \tag{8}$$

8. By setting $e = \dfrac{c}{GM}$ and $a = \dfrac{h^2}{GM(1-e^2)}$, show that (8) is equivalently formulated by the polar equation (3). Thus, the position (7) revolves around an elliptical orbit on the plane with major axis vector $\mathbf{i}$ and minor axis vector $\mathbf{j}$.