Exercises for class discussion.

1. Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ be an ellipsoid. Show that the tangent plane at the point (x_0, y_0, z_0) on the ellipsoid can be expressed by

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = 1$$

(Section 14.6: Exercise 57)

2. Suppose that a surface is determined by the implicit function $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{c}$.
   (a) Find the tangent plane at the point (x_0, y_0, z_0) on the surface.
   (b) Find the x-, y-, and z-intercepts of the tangent plane.
   (c) Show that the sum of the x-, y-, and z-intercepts of the tangent plane is determined by c, and therefore, that the sum is a constant (Section 14.6: Exercise 67).

Exercises performed by Maxima/wxMaxima. You are encouraged to work with Maxima/wxMaxima on your own computer. Lecture notes and wxMaxima examples from Section 14.6 to 14.8 are posted at iLearn. Work through examples presented at the lecture notes, and familiarize yourself with wxMaxima commands.

1. Find the directional derivative at $x = 2$, $y = 1$ in the direction $[-1, 2]$ for $f(x, y) = \log(x^2 + y^2)$.

2. Find an equation of the tangent plane at $x = 0$, $y = 0$, $z = 1$ for $F(x, y, z) = \log(x + z) - yz = 0$.

3. Find the local minimum and maximum and saddle points for $f(x, y) = x^3y + 12x^2 - 8y$. Graph the function that reveals all the aspects of critical point.

4. Find the local minimum and maximum and saddle points for $f(x, y) = 2x^3 + xy^2 + 5x^2 + y^2$. Graph the function that reveals all the aspects of critical point.

5. Apply Lagrange multipliers to find the minimum or the maximum for $f(x, y) = e^{xy}$ subject to $g(x, y) = x^3 + y^3 = 16$.

6. Apply Lagrange multipliers to find the minimum and the maximum for $f(x, y, z) = 8x - 4z$ subject to $g(x, y, z) = x^2 + 10y^2 + z^2 = 5$.

Online quiz No.5: Start online quiz at the course website.

Key formulas to memorize:

1. $\nabla f = \langle f_x(x_0, y_0), f_y(x_0, y_0)\rangle$ [the gradient vector at (x_0, y_0)]
   $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$ [the directional derivative at (x_0, y_0) in the direction $\mathbf{u}$ of unit vector]

2. $\nabla F = \langle F_x(x_0, y_0, z_0), F_y(x_0, y_0, z_0), F_z(x_0, y_0, z_0)\rangle$ [the gradient vector at (x_0, y_0, z_0)]
   $\nabla F \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$ [the tangent plane for the implicit function $F(x, y, z) = 0$]

3. Second derivative test with $D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2$ at a critical point (x_0, y_0) [that is, $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$]

$D > 0$ and $f_{xx}(x_0, y_0) > 0$	$f(x_0, y_0)$ is a local minimum
$D > 0$ and $f_{xx}(x_0, y_0) < 0$	$f(x_0, y_0)$ is a local maximum
$D < 0$	$f(x_0, y_0)$ is a saddle point

4. Method of Lagrange multipliers: The solution to

$$\nabla f = \lambda \nabla g \text{ (Lagrange multiplier) and } g(x, y, z) = k$$

maximizes or minimizes $w = f(x, y, z)$ subject to $g(x, y, z) = k$.