Discrete Distributions

A simplest example of random experiment is a coin-tossing, formally called "Bernoulli trial." It happens to be the case that many useful distributions are built upon this simplest form of experiment, whose relations are summarized later in a diagram.

Bernoulli trials. A Bernoulli random variable X takes value only on 0 and 1. It is determined by the **parameter** p (which represents the probability that $X = 1$), and the frequency function is given by

$$\begin{aligned} p(1) &= p \\ p(0) &= 1 - p \end{aligned}$$

If A is the event that an experiment results in a "success," then the **indicator random variable**, denoted by I_A, takes the value 1 if A occurs and the value 0 otherwise.

$$I_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A; \\ 0 & \text{otherwise (i.e., } \omega \notin A). \end{cases}$$

Then I_A is a Bernoulli random variable with "success" probability $p = P(A)$. We will call such experiment a **Bernoulli trial**.

Binomial distribution. If we have n independent Bernoulli trials, each with a success probability p, then the probability that there will be exactly k successes is given by

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots, n.$$

The above frequency function $p(k)$ is called a **binomial distribution** with parameter (n, p).

Example 1. Five fair coins are flipped independently. Find the frequency function of the number of heads obtained.

Solution. The number X of heads represents a binomial random variable with parameter $n = 5$ and $p = \frac{1}{2}$. Thus, we obtain

$$p(k) = \binom{5}{k} \left(\frac{1}{2}\right)^5$$

Example 2. A company has known that their screws is defective with probability 0.01. They sell the screws in packages of 10, and are planning a money-back guarantee

(a) at most one of the 10 screws is defective, and they replace it if a customer find more than one defective screws, or

(b) they replace it even if there is only one defective.

For each of the money-back guarantee plans above what proportion of packages sold must be replaced?

Solution. The number X of defective screws in a package represents a binomial random variable with parameter $p = 0.01$ and $n = 10$.

(a) $P(X \geq 2) = 1 - p(0) - p(1) = 1 - \binom{10}{0}(0.01)^0(0.99)^{10} - \binom{10}{1}(0.01)^1(0.99)^9 \approx 0.004$

(b) $P(X \geq 1) = 1 - p(0) = 1 - \binom{10}{0}(0.01)^0(0.99)^{10} \approx 0.096$

Properties of binomial random distributions. By applying the binomial theorem, we can immediately see the following results:

$$\sum_{k=0}^{n} p(k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = [p + (1-p)]^n = 1;$$
$$M(t) = \sum_{k=0}^{n} e^{kt} p(k) = \sum_{k=0}^{n} \binom{n}{k} (pe^t)^k (1-p)^{n-k} = [pe^t + (1-p)]^n.$$

Moments of binomial random distributions. The mgf $M(t)$ allows us to calculate the moments $E[X]$ and $E[X^2]$.

$$\begin{aligned} M'(t) &= \frac{d}{dt}\left([pe^t + (1-p)]^n\right) = npe^t[pe^t + (1-p)]^{n-1} \\ M''(t) &= \frac{d}{dt}\left(npe^t[pe^t + (1-p)]^{n-1}\right) \\ &= npe^t[pe^t + (1-p)]^{n-1} + n(n-1)(pe^t)^2[pe^t + (1-p)]^{n-2} \end{aligned}$$

Thus, we obtain $E[X] = M'(0) = np$ and $E[X^2] = M''(0) = np + n(n-1)p^2 = np + (np)^2 - np^2$. This allows us to calculate $\mathrm{Var}(X) = E[X^2] - (E[X])^2 = np - np^2 = np(1-p)$.

Relation between Bernoulli trials and binomial random variable. A binomial random variable can be expressed in terms of n Bernoulli random variables. If $X_1, X_2, \ldots, X_n$ are independent Bernoulli random variables with success probability p, then the sum of those random variables

$$Y = \sum_{i=1}^{n} X_i$$

is **distributed as** the binomial distribution with parameter (n, p).

Their relationship in mgf. Let X be a Bernoulli random variable with success probability p. Then the mgf $M_X(t)$ is given by

$$M_X(t) = e^{(0)t} \times (1-p) + e^{(1)t} \times p = (1-p) + pe^t.$$

Then we can derive the mgf of a binomial random variable $Y = \sum_{i=1}^{n} X_i$ from the mgf of Bernoulli random variable by applying the mgf property for independent random variables repeatedly.

$$M_Y(t) = M_{X_1}(t) \times M_{X_2}(t) \times \cdots \times M_{X_n}(t) = [(1-p) + pe^t]^n.$$

Sum of two independent random variables.

Theorem 3. *If X and Y are independent binomial random variables with respective parameters (n, p) and (m, p), then the sum $Z = X + Y$ is distributed as the binomial distribution with parameter $(n + m, p)$.*

Solution. By the mgf property for independent random variables we obtain

$$\begin{aligned} M_Z(t) &= M_X(t) \times M_Y(t) = [(1-p) + pe^t]^n \times [(1-p) + pe^t]^m \\ &= [(1-p) + pe^t]^{n+m} \end{aligned}$$

This uniquely determines the distribution for Z, which is binomial with parameter $(n + m, p)$.

Alternative approach to the proof. Observe that we can express

$$X = \sum_{i=1}^{n} Z_i \text{ and } Y = \sum_{i=n+1}^{n+m} Z_i$$

in terms of independent Bernoulli random variables Z_i's with success probability p. Thus, the resulting sum

$$X + Y = \sum_{i=1}^{n+m} Z_i$$

must be a binomial random variable with parameter $(n + m, p)$.

Expectations of Bernoulli and binomial random variables. Let X be a Bernoulli random variable with success probability p. Then the expectation of X becomes

$$E[X] = 0 \times (1-p) + 1 \times p = p.$$

Now let Y be a binomial random variable with parameter (n, p). Recall that Y can be expressed as the sum $\sum_{i=1}^{n} X_i$ of independent Bernoulli random variables $X_1, \ldots, X_n$ with success probability p. Thus, by using property (c) of expectation we obtain

$$E[Y] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = np.$$

Variances of Bernoulli and binomial random variables. Let X be a Bernoulli random variable with success probability p. Then the expectation $E[X]$ is p, and the variance of X is

$$\text{Var}(X) = (0-p)^2 \times (1-p) + (1-p)^2 \times p = p(1-p).$$

Since a binomial random variable Y is the sum $\sum_{i=1}^{n} X_i$ of independent Bernoulli random variables, we obtain

$$\text{Var}(Y) = \text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{Var}(X_i) = np(1-p).$$

Hypergeometric distribution. Consider the collection of N subjects, of which m belongs to one particular class (say, "tagged" subjects), and $(N-m)$ to another class (say, "non-tagged"). Now a sample of size r is chosen randomly from the collection of N subjects. Then the number X of "tagged" subjects selected has the frequency function

$$p(k) = \frac{\binom{m}{k}\binom{N-m}{r-k}}{\binom{N}{r}}$$

where $0 \le k \le r$ must also satisfy $k \ge r-(N-m)$ and $k \le m$. The above frequency function $p(k)$ is called a **hypergeometric** distribution with parameter (N, m, r).

Example 4. A lot, consisting of 50 items, is inspected. Suppose that the lot contains 5 defective items. In the following two inspection procedures, What is the probability that no defective items are found in the inspection?

(a) Four items are randomly selected and tested.

(b) Ten items are randomly selected and tested.

Solution. The number X of defective items in the inspection has a hypergeometric distribution with $N = 50$, $m = 5$.

(a) Here we choose $r = 4$, and calculate $P(X=0) = \frac{\binom{5}{0}\binom{45}{4}}{\binom{50}{4}} \approx 0.65$

(b) Here we choose $r = 10$, and calculate $P(X=0) = \frac{\binom{5}{0}\binom{45}{10}}{\binom{50}{10}} \approx 0.31$

Relation between Bernoulli trials and Hypergeometric distribution. Let A_i be the event that a "tagged" subject is found at the i-th selection.

$$X_i = \begin{cases} 1 & \text{if } A_i \text{ occurs;} \\ 0 & \text{otherwise.} \end{cases}$$

Then X_i is a Bernoulli trial with "success" probability $p = P(A_i) = \frac{m}{N}$. Then the number Y of "tagged" subjects in a sample of size r can be expressed in terms of r Bernoulli random variables.

$$Y = \sum_{i=1}^{r} X_i$$

is **distributed as** a hypergeometric distribution with parameter (N, m, r).

Example 5. An urn contains 10 red balls and 20 blue balls, and balls are drawn one at a time without replacement. Let A_i be the event that a red ball is drawn at the i-th attempt.

(a) Find $P(A_1)$

(b) Calculate $P(A_2)$.

(c) Find $P(A_i)$ in general.

Solution.

(a) $P(A_1) = \dfrac{10}{30} = \dfrac{1}{3}$.

(b) If we draw 2 balls then we have 10×29 outcomes in which the second ball is red. Thus, $P(A_2) = \dfrac{10 \times 29}{30 \times 29} = \dfrac{1}{3}$.

(c) In general we have $10 \times 29 \times 28 \times \cdots \times (31 - i)$ outcomes in which i-th ball is red, and obtain

$$P(A_i) = \frac{10 \times 29 \times 28 \times \cdots \times (31 - i)}{30 \times 29 \times 28 \times \cdots \times (31 - i)} = \frac{1}{3}.$$

Expectation of hypergeometric random variable. Let X_i be a Bernoulli trial of finding a "tagged" subject in the i-th selection. Then the expectation of X becomes

$$E[X_i] = \frac{m}{N}.$$

Now let Y be a hypergeometric random variable with parameter (N, m, r). Recall that Y can be expressed as the sum $\sum_{i=1}^{r} X_i$ of the above Bernoulli trials. We can easily calculate

$$E[Y] = E\left[\sum_{i=1}^{r} X_i\right] = \sum_{i=1}^{r} E[X_i] = \frac{mr}{N}$$

Dependence of Bernoulli trials in Hypergeometric distribution. Suppose that $i \neq j$. Then we can find that

$$X_i X_j = \begin{cases} 1 & \text{if } A_i \text{ and } A_j \text{ occurs;} \\ 0 & \text{otherwise.} \end{cases}$$

is again a Bernoulli trial with "success" probability $p = P(A_i \cap A_j) = \frac{m(m-1)}{N(N-1)}$. Since $E[X_i] = E[X_j] = \frac{m}{N}$ and $E[X_i X_j] = \frac{m(m-1)}{N(N-1)}$, we can calculate

$$\mathrm{Cov}(X_i, X_j) = E[X_i X_j] - E[X_i]E[X_j] = \frac{m(m-N)}{N^2(N-1)} < 0$$

Therefore, X_i and X_j are dependent, and negatively correlated.

Variance of hypergeometric random variable. The variance of Bernoulli trial with success probability $\frac{m}{N}$ is given by

$$\mathrm{Var}(X_i) = \left(\frac{m}{N}\right)\left(1 - \frac{m}{N}\right) = \frac{m(N-m)}{N^2}$$

Together with $\mathrm{Cov}(X_i, X_j) = \frac{m(m-N)}{N^2(N-1)}$, we can calculate

$$\begin{aligned}\mathrm{Var}(Y) &= \mathrm{Var}\left(\sum_{i=1}^{r} X_i\right) = \sum_{i=1}^{r} \mathrm{Var}(X_i) + 2\sum_{i<j} \mathrm{Cov}(X_i, X_j) \\ &= r \times \frac{m(N-m)}{N^2} + r(r-1) \times \frac{m(m-N)}{N^2(N-1)} \\ &= \frac{mr(N-m)(N-r)}{N^2(N-1)}\end{aligned}$$

Example 6. We capture 30 individuals, mark them with a yellow tag, and then release them back into the environment. Later we capture another sample of 20 individuals. Some of the individuals in this sample have been marked, known as "recaptures." Let X be the number of recaptures. Suppose that the population size N is unknown.

(a) What is the frequency function for X?

(b) How can we guess the population size N from the observed number X of recaptures?

Solution.

(a) It is hypergeometric with $m = 30$ and $r = 20$, which gives $p(k) = \frac{\binom{30}{k}\binom{N-30}{20-k}}{\binom{N}{r}}$ for $k = 0, 1, \ldots, 20$, assuming that $N \geq 50$.

(b) Since the best guess for X is the mean value $E[X] = \frac{mr}{N} = \frac{600}{N}$, we can form the equation $X \approx \frac{600}{N}$ in approximation. By solving it, we can estimate $N \approx \frac{600}{X}$ if $X \geq 1$. If $X = 0$, there is no guess for N.

Geometric series formulas. For $0 < a < 1$, we can establish the following formulas.

$$\sum_{k=1}^{n} a^{k-1} = \frac{1-a^n}{1-a}; \qquad \sum_{k=1}^{\infty} a^{k-1} = \frac{1}{1-a}; \tag{3.1}$$

$$\sum_{k=1}^{\infty} k a^{k-1} = \frac{d}{dx}\left(\sum_{k=0}^{\infty} x^k\right)\Big|_{x=a} = \frac{d}{dx}\left(\frac{1}{1-x}\right)\Big|_{x=a} = \frac{1}{(1-a)^2}; \tag{3.2}$$

$$\begin{aligned}\sum_{k=1}^{\infty} k^2 a^{k-1} &= \frac{d}{dx}\left(\sum_{k=1}^{\infty} k x^k\right)\Big|_{x=a} = \frac{d}{dx}\left(\frac{x}{(1-x)^2}\right)\Big|_{x=a} \\ &= \frac{1}{(1-a)^2} + \frac{2a}{(1-a)^3}\end{aligned}$$

Geometric random variables. In sampling independent Bernoulli trials, each with probability of success p, the frequency function for the number of trials at the first success is

$$p(k) = (1-p)^{k-1}p \qquad k = 1, 2, \ldots.$$

This is called a *geometric distribution*. A random variable X is called a *geometric random variable* if it has a frequency function

$$P(X = k) = (1-p)^{k-1}p$$

for $k = 1, 2, \ldots$.

Example 7. Consider a roulette wheel consisting of 38 numbers—1 through 36, 0, and double 0. If Mr.Smith always bets that the outcome will be one of the numbers 1 through 12, what is the probability that (a) his first win will occur on his fourth bet; (b) he will lose his first 5 bets?

Solution. The number X of bets until the first win is a geometric random variable with success probability $p = \left(\frac{12}{38}\right) = \left(\frac{6}{19}\right)$.

$$\text{(a) } P(X = 4) = \left(\frac{13}{19}\right)^3 \left(\frac{6}{19}\right) \approx 0.1$$

$$\text{(b) } P(X \geq 6) = \sum_{k=6}^{\infty} (1-p)^{k-1}p$$

$$= (1-p)^5 p \sum_{j=1}^{\infty} (1-p)^{j-1} = (1-p)^5 = \left(\frac{13}{19}\right)^5 \approx 0.15$$

$\{X \geq 6\} = \{\text{He loses the first five bets}\}$; thus, $P(X \geq 6) = \left(\frac{13}{19}\right)^5$.

Properties of geometric distribution By using the geometric series formula (3.1) we can immediately see the following:

$$\sum_{k=1}^{\infty} p(k) = p \sum_{k=1}^{\infty} (1-p)^{k-1} = \frac{p}{1-(1-p)} = 1;$$

$$M(t) = \sum_{k=1}^{\infty} e^{kt} p(k) = pe^t \sum_{k=1}^{n} [(1-p)e^t]^{k-1} = \frac{pe^t}{1-(1-p)e^t}.$$

Expectation and variance of geometric distribution. Let X be a geometric random variable with success probability p. To calculate the expectation $E[X]$ and the variance $\mathrm{Var}(X)$, we apply the geometric series formula (3.2) to obtain

$$E[X] = \sum_{k=1}^{\infty} kp(k) = p \sum_{k=1}^{\infty} k(1-p)^{k-1} = \frac{1}{p};$$

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = \sum_{k=1}^{\infty} k^2 p(k) - \left(\frac{1}{p}\right)^2$$

$$= p\left(\frac{1}{p^2} + \frac{2(1-p)}{p^3}\right) - \frac{1}{p^2} = \frac{1-p}{p^2}.$$

Moments of geometric distribution. We can calculate $M'(t)$ and $M''(t)$ for $M(t) = \frac{pe^t}{1-(1-p)e^t}$.

$$M'(t) = \frac{pe^t}{(1-(1-p)e^t)^2}$$
$$M''(t) = \frac{d}{dt}\left[\frac{pe^t}{(1-(1-p)e^t)^2}\right] = \frac{pe^t + p(1-p)e^{2t}}{(1-(1-p)e^t)^3}$$

Then we obtain $E[X] = M'(0) = \frac{1}{p}$, and $E[X^2] = M''(0) = \frac{2-p}{p^2}$. Thus, we have

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = \frac{1-p}{p^2}$$

Negative binomial distribution. Given independent Bernoulli trials with probability of success p, the frequency function of the number of trials until the r-th success is

$$p(k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}, \quad k = r, r+1, \ldots.$$

This is called a *negative binomial distribution* with parameter (r, p).

Example 8. A fair coin is flipped repeatedly until heads are observed four times. Find the frequency function of the number of attempts to flip a coin.

Solution. The number X of attempts represents a negative binomial random variable with parameter $r = 4$ and $p = \frac{1}{2}$. Thus, we obtain

$$p(k) = \binom{k-1}{3} \left(\frac{1}{2}\right)^k, \quad k = 4, 5, 6, \ldots$$

Relation between geometric and negative binomial distribution. The geometric distribution is a special case of negative binomial distribution when $r = 1$. Moreover, if $X_1, \ldots, X_r$ are independent and identically distributed (iid) geometric random variables with parameter p, then the sum

$$Y = \sum_{i=1}^{r} X_i \tag{3.3}$$

becomes a negative binomial random variable with parameter (r, p).

Expectation, variance and mgf of negative binomial distribution. By using the sum of iid geometric random variables in (3.3), we can compute the expectation, the variance, and the mgf of negative binomial random variable Y.

$$E[Y] = E\left[\sum_{i=1}^{r} X_i\right] = \sum_{i=1}^{r} E[X_i] = \frac{r}{p};$$
$$\mathrm{Var}(Y) = \mathrm{Var}\left(\sum_{i=1}^{r} X_i\right) = \sum_{i=1}^{r} \mathrm{Var}(X_i) = \frac{r(1-p)}{p^2};$$
$$M_Y(t) = M_{X_1}(t) \times M_{X_2}(t) \times \cdots \times M_{X_r}(t) = \left[\frac{pe^t}{1-(1-p)e^t}\right]^r.$$

Example 9. What is the average number of times one must throw a die until the outcome "1" has occurred 4 times?

Solution. The number X of throws until the fourth outcome of "1" is recorded is a negative binomial random variable with $r = 4$ and $p = \frac{1}{6}$. Thus, we obtain $E[X] = \frac{r}{p} = 24$.

Euler's number and natural exponential. The number

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = 2.7182\cdots$$

is known as the base of the natural logarithm (or, called Euler's number). The exponential function $f(x) = e^x$ is associated with the Taylor series

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}. \tag{3.4}$$

Poisson distribution. The **Poisson distribution** with parameter λ ($\lambda > 0$) has the frequency function

$$p(k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots$$

By applying the Taylor series (3.4) we can immediately see the following:

$$\sum_{k=0}^{\infty} p(k) = e^{-\lambda}\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda}e^{\lambda} = 1.$$

A Poisson random variable X represents the number of successes when there are a large (and usually unknown) number of independent trials.

Example 10. Suppose that the number of typographical errors per page has a Poisson distribution with parameter $\lambda = 0.5$.

(a) Find the probability that there are no errors on a single page.

(b) Find the probability that there is at least one error on a single page.

Solution.

(a) $P(X = 0) = p(0) = e^{-0.5} \approx 0.607$

(b) $P(X \geq 1) = 1 - p(0) = 1 - e^{-0.5} \approx 0.393$

Mgf and moments of Poisson distribution. By applying the Taylor series (3.4) we obtain

$$M(t) = \sum_{k=0}^{\infty} e^{kt} p(k) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(e^t \lambda)^k}{k!} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)}.$$

Then we can calculate $M'(t) = \lambda e^t e^{\lambda(e^t-1)}$ and $M''(t) = (\lambda e^t)^2 e^{\lambda(e^t-1)} + \lambda e^t e^{\lambda(e^t-1)}$. Thus, we obtain $E[X] = M'(0) = \lambda$ and $E[X^2] = M''(0) = \lambda^2 + \lambda$. Finally we get $\mathrm{Var}(X) = E[X^2] - (E[X])^2 = \lambda$.

Poisson approximation to binomial distribution. The Poisson distribution can be used as an approximation for a binomial distribution with parameter (n, p) when n is large and p is small enough so that np is a moderate number λ. Let $\lambda = np$. Then the binomial frequency function becomes

$$\begin{aligned} p(k) &= \frac{n!}{k!(n-k)!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \\ &= \frac{\lambda^k}{k!} \cdot \frac{n!}{(n-k)! n^k} \cdot \left(1 - \frac{\lambda}{n}\right)^n \cdot \left(1 - \frac{\lambda}{n}\right)^{-k} \\ &\to \frac{\lambda^k}{k!} \cdot 1 \cdot e^{-\lambda} \cdot 1 = e^{-\lambda} \frac{\lambda^k}{k!} \quad \text{as } n \to \infty. \end{aligned}$$

Example 11. Suppose that the a microchip is defective with probability 0.02. Find the probability that a sample of 100 microchips will contain at most 1 defective microchip.

Solution. The number X of defective microchips has a binomial distribution with $n = 100$ and $p = 0.02$. Thus, we obtain

$$\begin{aligned} P(X \le 1) &= p(0) + p(1) \\ &= \binom{100}{0} (0.98)^{100} + \binom{100}{1} (0.02)(0.98)^{99} \approx 0.403 \end{aligned}$$

Approximately X has a Poisson distribution with $\lambda = (100)(0.02) = 2$. We can calculate

$$P(X \le 1) = p(0) + p(1) = e^{-2} + (2)(e^{-2}) \approx 0.406$$

Expectation and variance of Poisson distribution. Let X_n, $n = 1, 2, \ldots$, be a sequence of binomial random variables with parameter $(n, \lambda/n)$. Then we regard the limit of the sequence as a Poisson random variable Y, and use the limit to find $E[Y]$ and $\mathrm{Var}(Y)$. Thus, we obtain

$$\begin{aligned} E[X_n] = \lambda \quad &\to \quad E[Y] = \lambda \quad \text{as } n \to \infty; \\ \mathrm{Var}(X_n) = \lambda \left(1 - \frac{\lambda}{n}\right) \quad &\to \quad \mathrm{Var}(Y) = \lambda \quad \text{as } n \to \infty. \end{aligned}$$

Sum of two independent random variables.

Theorem 12. *The sum of independent Poisson random variables is a Poisson random variable: If X and Y are independent Poisson random variables with respective parameters λ_1 and λ_2, then $Z = X + Y$ is distributed as the Poisson distribution with parameter $\lambda_1 + \lambda_2$.*

Solution. By the mgf property for independent random variables we obtain

$$M_Z(t) = M_X(t) \times M_Y(t) = e^{\lambda_1(e^t-1)} \times e^{\lambda_2(e^t-1)} = e^{(\lambda_1+\lambda_2)(e^t-1)}$$

This uniquely determines the distribution for Z, which is Poisson with parameter $\lambda_1 + \lambda_2$.

Alternative approach to the proof. Let X_n and Y_n be independent binomial random variables with respective parameter (k_n, p_n) and (l_n, p_n). Then the sum $X_n + Y_n$ of the random variables has the binomial distribution with parameter $(k_n + l_n, p_n)$. By letting $k_n p_n \to \lambda_1$ and $l_n p_n \to \lambda_2$ with $k_n, l_n \to \infty$ and $p_n \to 0$, the respective limits of X_n and Y_n are Poisson random variables X and Y with respective parameters λ_1 and λ_2. Meanwhile, the limit of $X_n + Y_n$ is the sum of the random variables X and Y, and has a Poisson distribution with parameter $\lambda_1 + \lambda_2$.

Example 13. Suppose that at an office the average number of calls per hour is 2. (a) If the number X of calls for the first hour has a Poisson distribution, what is the parameter λ? (b) Let Y be the number of calls for three hours. What is the distribution for Y? what is the value for parameter? (c) What is the probability that the first call comes after 3 hours? (d) In general what is the probability that the first call comes after t hours?

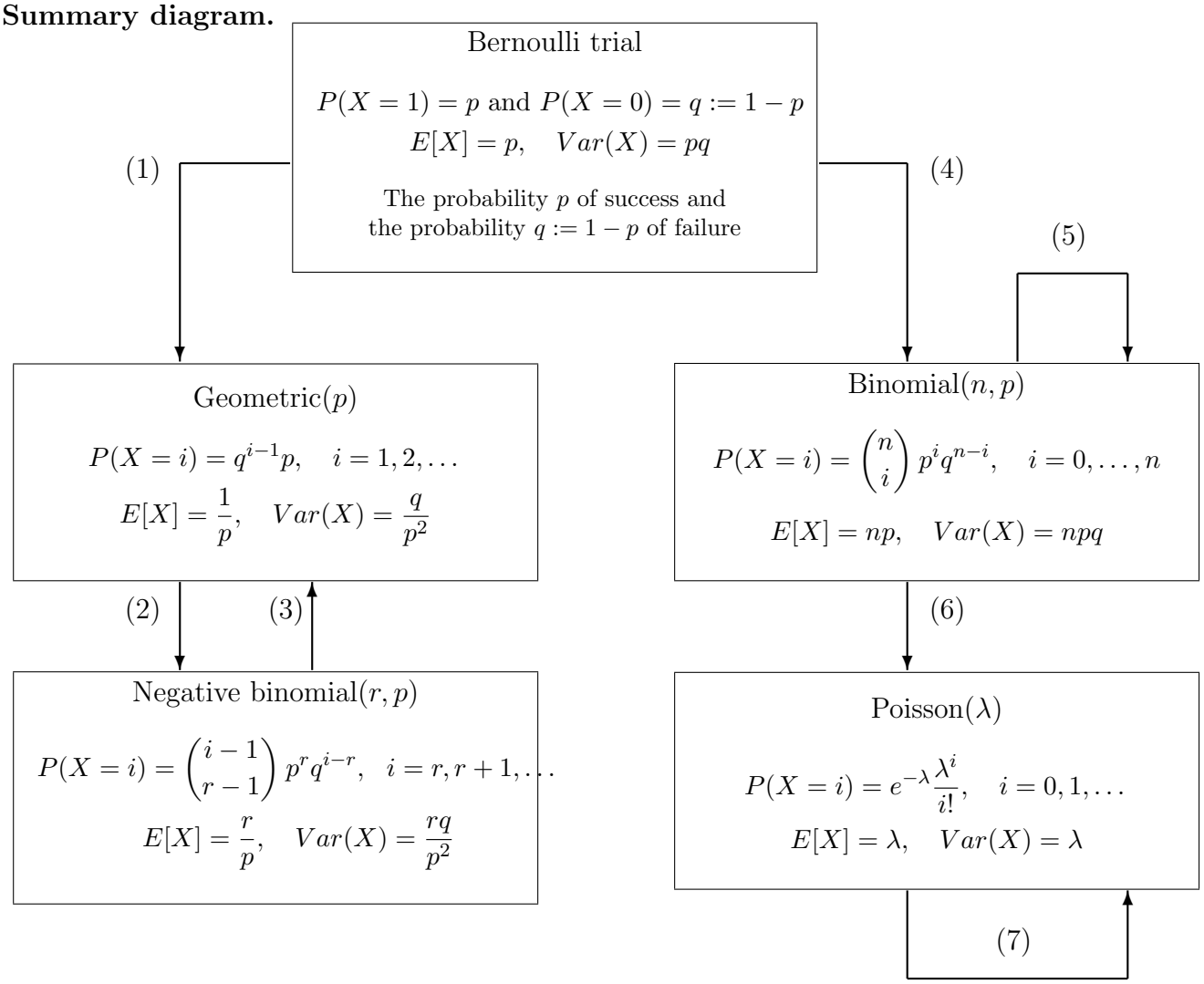
Solution. (a) Since $E[X] = \lambda$, it must be $\lambda = 2$.

(b) Let X_i be the number of calls during the i-th hour. Then $Y = X_1 + X_2 + X_3$ must have a Poisson distribution with parameter $\lambda = (2)(3) = 6$.

(c) The probability is obtained as $P(Y = 0) = e^{-6} \approx 0.002$.

(d) Let Y be the number of calls for t hours. Then the probability is obtained as $P(Y = 0) = e^{-2t}$.

Summary diagram.



Relationship among random variables.

(1) X is the number of Bernoulli trials until the *first* success occurs.

(2) X is the number of Bernoulli trials until the rth success occurs. If $X_1, \cdots, X_r$ are independent geometric random variables with parameter p, then $X = \sum_{k=1}^{r} X_k$ becomes a negative binomial random variable with parameter (r, p).

(3) $r = 1$.

(4) X is the number of successes in n Bernoulli trials. If $X_1, \cdots, X_n$ are independent Bernoulli trials, then $X = \sum_{k=1}^{n} X_k$ becomes a binomial random variable with (n, p).

(5) If X_1 and X_2 are independent binomial random variables with respective parameters (n_1, p) and (n_2, p), then $X = X_1 + X_2$ is *also* a binomial random variable with $(n_1 + n_2, p)$.

Relationship among random variables, continued.

(6) Poisson approximation is used for binomial distribution by letting $n \to \infty$ and $p \to 0$ while $\lambda = np$.

(7) If X_1 and X_2 are independent Poisson random variables with respective parameters λ_1 and λ_2, then $X = X_1 + X_2$ is *also* a Poisson random variable with $\lambda_1 + \lambda_2$.

Homework

Problem 1. A study shows that 40% of college students binge drink. Let X be the number of students who binge drink out of sample size $n = 12$.

(a) Find the mean and standard deviation of X.

(b) Do you agree that the probability that X is 5 or less is higher than 50%? Justify your answer.

Problem 2. In a lot of 20 light bulbs, there are 9 bad bulbs. Let X be the number of defective bulbs found in the inspection. Find the frequency function $p(k)$, and identify the range for k in the following inspection procedures.

(a) An inspector inspects 5 bulbs selected at random and without replacement.

(b) An inspector inspects 15 bulbs selected at random and without replacement.

Problem 3. Appending three extra bits to a 4-bit word in a particular way (a Hamming code) allows detection and correction of up to one error in any of the bits. (a) If each bit has probability .05 of being changed during communication, and the bits are changed independently of each other, what is the probability that the word is correctly received (that is, 0 or 1 bit is in error)? (b) How does this probability compare to the probability that the word will be transmitted correctly with no check bits, in which case all four bits would have to be transmitted correctly for the word to be correct?

Problem 4. Which is more likely: 9 heads in 10 tosses of a fair coin or 18 heads in 20 tosses?

Problem 5. Two boys play basketball in the following way. They take turns shooting and stop when a basket is made. Player A goes first and has probability p_1 of making a basket on any throw. Player B goes, who shoots second, has probability p_2 of making a basket. The outcomes of the successive trials are assumed to be independent.

(a) Find the frequency function for the total number of attempts.

(b) What is the probability that player A wins?

Problem 6. Suppose that in a sequence of independent Bernoulli trials each with probability of success p, the number of failure up to the first success is counted. (a) What is the frequency function for this random variable? (b) Find the frequency function for the number of failures up to the rth success.

Hint: In (b) let X be the number of failures up to the rth success, and let Y be the number of trials until the r-th success. What is the relation between the random variables X and Y?

Problem 7. Find an expression for the cumulative distribution function of a geometric random variable.

Problem 8. In a sequence of independent trials with probability p of success, what is the probability that there are *exactly* r successes before the kth failure?

Problem 9. The probability that a man in a certain age group dies in the next four years is $p = 0.05$. Suppose that we observe 20 men.

(a) Find the probability that two of them or fewer die in the next four years.

(b) Find an approximation by using a Poisson distribution.

Problem 10. The probability of being dealt a royal straight flush (ace, king, queen, jack, and ten of the same suit) in poker is about 1.5×10^{-6}. Suppose that an avid poker player sees 100 hands a week, 52 weeks a year, for 20 years.

(a) What is the probability that she never sees a royal straight flush dealt?

(b) What is the probability that she sees at least two royal straight flushes dealt?

Problem 11. Professor Rice was told that he has only 1 chance in 10,000 of being trapped in a much-maligned elevator in the mathematics building. Assume that the outcomes on all the days are mutually independent. If he goes to work 5 days a week, 52 weeks a year, for 10 years and always rides the elevator up to his office when he first arrives.

(a) What is the probability that he will never be trapped?

(b) What is the probability that he will be trapped once?

(c) What is the probability that he will be trapped twice?

Problem 12. Suppose that a rare disease has an incidence of 1 in 1000. and that members of the population are affected independently. Let X be the number of cases in a population of $100,000$.

(a) What is the average number of cases?

(b) Find the frequency function.

Problem 13. Suppose that in a city the number of suicides can be approximated by a Poisson random variable with $\lambda = 0.33$ per month.

(a) What is the probability of two suicides in January?

(b) What is the distribution for the number of suicides per year?

(c) What is the average number of suicides in one year?

(d) What is the probability of two suicides in one year?

Optional problems

Problem 14. If n men throw their hats into a pile and each man takes a hat at random, what is the expected number of matches?
Hint: Suppose Y is the number of matches. How can you express the random variable Y by using indicator random variables $X_1, \ldots, X_n$? What is the event for the indicator random variable X_i?

Problem 15. Suppose that n enemy aircraft are shot at simultaneously by m gunners, that each gunner selects an aircraft to shoot at independently of the other gunners, and that each gunner hits the selected aircraft with probability p. Find the expected number of aircraft hit by the gunners.
Hint: For each $i = 1, \ldots, n$, let X_i be the indicator random variable of the event that the i-th aircraft was hit. Explain how you can verify $P(X_i = 0) = \left(1 - \frac{p}{n}\right)^m$ by yourself, and use it to find the expectation of interest.

Problem 16. (Banach Match Problem) A pipe smoker carries one box of matches in his left pocket and one in his right. Initially, each box contains n matches. If he needs a match, the smoker is equally likely to choose either pocket. What is the frequency function for the number of matches in the other box when he first discovers that one box is empty?

Hint: Let A_k denote the event that the pipe smoker first discovers that the right-hand box is empty and that there are k matches in the left-hand box at the time, and similarly let B_k denote the event that he first discovers that the left-hand box is empty and that there are k matches in the right-hand box at the time. Then use A_k's and B_k's to express the frequency function of interest.

Problem 17. Let X be a negative binomial random variable with parameter (r, p).

(a) When $r = 2$ and $p = 0.2$, find the probability $P(X \leq 4)$.

(b) Let Y be a binomial random variable with parameter (n, p). When $n = 4$ and $p = 0.2$, find the probability $P(Y \geq 2)$.

(c) By comparing the results above, what can you find about the relationship between $P(X \leq 4)$ and $P(Y \geq 2)$? Generalize the relationship to the one between $P(X \leq n)$ and $P(Y \geq 2)$, and find the probability $P(X \leq 10)$ when $r = 2$ and $p = 0.2$.

(d) By using the Poisson approximation, find the probability $P(X \leq 1000)$ when $r = 2$ and $p = 0.001$.

Answers to exercises

Problem 1.

(a) The mean is $np = (12)(0.4) = 4.8$, and the standard deviation is $\sqrt{(12)(0.4)(0.6)} \approx 1.7$.

(b) Yes, because $P(X \le 5) \approx 0.665$.

Problem 2. X has a hypergeometric distribution with $N = 20$ and $m = 9$.

(a) Here we choose $r = 5$. Thus,

$$p(k) = \frac{\binom{9}{k}\binom{11}{5-k}}{\binom{20}{5}}$$

where k takes a value in the range of $0 \le k \le 5$.

(b) Here we choose $r = 15$. Thus,

$$p(k) = \frac{\binom{9}{k}\binom{11}{15-k}}{\binom{20}{15}}$$

where k takes a value in the range of $4 \le k \le 9$.

Problem 3. The number X of error bits has a binomial distribution with $n = 4$ and $p = 0.05$.

(a) $P(X \le 1) = \binom{4}{0}(0.95)^4 + \binom{4}{1}(0.05)(0.95)^3 \approx 0.986.$

(b) $P(X = 0) = \binom{4}{0}(0.95)^4 \approx 0.815.$

Problem 4. "9 heads in 10 tosses" $[\binom{10}{9}(1/2)^{10} \approx 9.77 \times 10^{-3}]$ is more likely than "18 heads in 20 tosses" $[\binom{20}{18}(1/2)^{20} \approx 1.81 \times 10^{-4}]$.

Problem 5. Let X be the total number of attempts.

(a) $P(X = n) = \begin{cases} (1-p_1)^{k-1}(1-p_2)^{k-1}p_1 & \text{if } n = 2k-1; \\ (1-p_1)^{k}(1-p_2)^{k-1}p_2 & \text{if } n = 2k. \end{cases}$

(b) $P(\{\text{Player A wins}\}) = \sum_{k=1}^{\infty} P(X = 2k-1) = p_1 \sum_{k=1}^{\infty}[(1-p_1)(1-p_2)]^{k-1} = \dfrac{p_1}{p_1 + p_2 - p_1 p_2}$

Problem 6. Let X be the number of failures up to the rth success, and let Y be the number of trials until the r-th success. Then Y has a negative binomial distribution, and has the relation $X = Y - r$.

(a) Here $r = 1$ (that is, Y has a geometric distribution), and $P(X = k) = P(Y = k+1) = (1-p)^k p$.

(b) $P(X = k) = P(Y = k + r) = \binom{k+r-1}{r-1} p^r (1-p)^k.$

Problem 7. Let X be a geometric random variable.

$$P(X \le k) = \sum_{i=1}^{k} p(1-p)^{i-1} = 1 - (1-p)^k$$

Problem 8. Let X be the number of trials up to the k-th failure. Then X has a negative binomial distribution with "success probability" $(1-p)$.

$$\begin{aligned} &P(\{\text{Exactly } r \text{ successes before the } k\text{-th failure}\}) \\ &= P(X = r+k) = \binom{k+r-1}{k-1}(1-p)^k p^r \end{aligned}$$

Problem 9.

(a) The number X of men dying in the next four years has a binomial distribution with $n = 20$ and $p = 0.05$.

$$\begin{aligned} &P(X \le 2) = p(0) + p(1) + p(2) \\ &= \binom{20}{0}(0.95)^{20} + \binom{20}{1}(0.05)(0.95)^{19} + \binom{20}{2}(0.05)^2(0.95)^{18} \\ &\approx 0.925 \end{aligned}$$

(b) The random variable X has approximately a Poisson distribution with $\lambda = (0.05)(20) = 1$.

$$P(X \le 2) = e^{-1} + e^{-1} + \frac{1}{2}e^{-1} = \frac{5}{2}e^{-1} \approx 0.920$$

Problem 10.

(a) The number X of royal straight flushes has a Poisson distribution with

$$\lambda = 1.5 \times 10^{-6} \times (100 \times 52 \times 20) = 0.156$$

Thus, we obtain

$$P(X = 0) = p(0) = e^{-0.156} \approx 0.856$$

(b) $P(X \ge 2) = 1 - p(0) - p(1) = 1 - e^{-0.156} - (0.156)(e^{-0.156}) = 1 - (1.156)(e^{-0.156}) \approx 0.011$

Problem 11.

(a) The number X of misfortunes for Professor Rice being trapped has a Poisson distribution with

$$\lambda = 10^{-4} \times (5 \times 52 \times 10) = 0.26$$

Thus, we obtain

$$P(X = 0) = p(0) = e^{-0.26} \approx 0.771$$

(b) $P(X = 1) = p(1) = (0.26)(e^{-0.26}) \approx 0.20$

(c) $P(X = 2) = p(2) = \frac{(0.26)^2}{2}(e^{-0.26}) \approx 0.026$

Problem 12.

(a) The number X of cases has a Poisson distribution with

$$\lambda = 10^{-3} \times 100,000 = 100$$

Thus, we obtain $E[X] = \lambda = 100$.

(b) $p(k) = (e^{-100})\dfrac{100^k}{k!}$ for $k = 0, 1, \ldots$.

Problem 13.

(a) Let X_1 be the number of suicides in January. Then we have

$$P(X_1 = 2) = \frac{(0.33)^2}{2}(e^{-0.33}) \approx 0.039$$

(b) Let $X_1,\ldots,X_{12}$ be the number of suicides in i-th month. Then

$$Y = X_1 + X_2 + \cdots + X_{12}$$

represents the number of suicides in one year, and it has a poisson distribution.

(c) $E[Y] = E[X_1] + E[X_2] + \cdots + E[X_{12}] = (12)(0.33) = 3.96$.

(d) Since $\lambda = E[Y] = 3.96$ is the parameter value for the Poisson random variable Y, we obtain

$$P(Y = 2) = \frac{(3.96)^2}{2}(e^{-3.96}) \approx 0.149$$