

Q.1 (2 points) Let $p(k)$ be the frequency function for a discrete random variable X taking values on $-1, 0$, or 1 . Suppose that $p(0) = \frac{1}{4}$, and that $p(-1)$ and $p(1)$ are unknown.

(a) Find $E[X^2]$.

$$E[X^2] = p(-1) + p(1) = 1 - p(0) = \frac{3}{4}$$

(b) If $E[X] = \frac{1}{4}$ then identify the value of $p(1)$.

$$E[X] = -p(-1) + p(1) = \frac{1}{4}. \text{ Together with } p(-1) + p(1) = \frac{3}{4}, \text{ we obtain } p(1) = \frac{1}{2}.$$

Q.2 (4 points) A pair (X, Y) of discrete random variables has the joint frequency function

$$p(x, y) = cxy, \quad x = 1, 2, 3 \text{ and } y = 1, 2$$

with constant c .

(a) Find the constant c .

$$\sum_{x=1}^3 \sum_{y=1}^2 p(x, y) = 18c = 1. \text{ Thus, we obtain } c = 1/18$$

(b) Find the marginal frequency function of X and Y .

$$p_X(x) = \sum_{y=1}^2 \frac{xy}{18} = \frac{x}{6} \text{ at } x = 1, 2, 3, \text{ and } p_Y(y) = \sum_{x=1}^3 \frac{xy}{18} = \frac{y}{3} \text{ at } y = 1, 2.$$

(c) Are X and Y independent?

Yes X and Y are independent, since the joint frequency function satisfies $p(x, y) = p_X(x)p_Y(y)$ for all $x = 1, 2, 3$ and $y = 1, 2$.

(d) Using $E[X] = \frac{7}{3}$, $E[Y] = \frac{5}{3}$, find $E[(1+X)(1+Y)]$.

By using $E[X] = \sum_{x=1}^3 (x) \left(\frac{x}{6}\right) = \frac{7}{3}$; $E[Y] = \sum_{y=1}^2 (y) \left(\frac{y}{3}\right) = \frac{5}{3}$, we obtain $E[1+X] = \frac{10}{3}$ and $E[1+Y] = \frac{8}{3}$. Since X and Y are independent, we find $E[(1+X)(1+Y)] = E[1+X] \cdot E[1+Y] = \frac{80}{9}$.

Q.3 (3 points) Let X and Y be random variables. The joint density function of X and Y is given by

$$f(x, y) = \begin{cases} x + y & \text{if } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1; \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $P(X \leq \frac{1}{2}, Y \leq \frac{1}{2})$.

$$P(X \leq \frac{1}{2}, Y \leq \frac{1}{2}) = \int_0^{1/2} \int_0^{1/2} (x + y) dx dy = \frac{1}{8}$$

(b) Find the marginal density function for X .

$$f_X(x) = \int_0^1 (x + y) dy = x + \frac{1}{2}.$$

(c) Find the conditional density function of Y given $X = x$.

$$f_{Y|X}(y|x) = \frac{x + y}{x + 1/2} \text{ for } 0 \leq y \leq 1.$$

Q.4 (1 points) Express $\text{Cov}(X + Y, X - Y)$ in terms of $\text{Var}(X)$ and $\text{Var}(Y)$.

Let $\mu_x = E[X]$ and $\mu_y = E[Y]$. Then we obtain

$$\begin{aligned}\text{Cov}(X + Y, X - Y) &= E[(X + Y - (\mu_x + \mu_y))(X - Y - (\mu_x - \mu_y))] \\ &= E[(X - \mu_x)^2 - (Y - \mu_y)^2] \\ &= E[(X - \mu_x)^2] - E[(Y - \mu_y)^2] \\ &= \text{Var}(X) - \text{Var}(Y)\end{aligned}$$