joint frequency/density function → marginal frequency/density function

$$E[g(X)] = \begin{cases} \sum_x g(x) p_X(x) \\ \int_{-\infty}^{\infty} g(x) f_X(x)\,dx \end{cases}$$

Q.1 (4 points) Let X be a random variable. We know that $E[X] = \frac{1}{4}$ and $E[X^2] = \frac{1}{2}$.

(a) Find $\mathrm{Var}(X)$.

$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = 7/16$.

(b) Find $E[2X+1]$ and $E[(X+1)^2]$.

$E[2X+1] = 2E[X] + 1 = \frac{3}{2}$ and $E[(X+1)^2] = E[X^2] + 2E[X] + 1 = 2$.

Q.2 (6 points) A pair (X, Y) of discrete random variables has the joint frequency function

$$p(x, y) = cxy, \ x = 1, 2, 3 \text{ and } y = 1, 2$$

with constant c.

(a) Find the constant c.

$\sum_{x=1}^{3} \sum_{y=1}^{2} p(x, y) = 18c = 1$. Thus, we obtain $c = 1/18$

(b) Find the marginal frequency function of X and Y. Are X and Y independent?

$p_X(x) = \sum_{y=1}^{2} \frac{xy}{18} = \frac{x}{6}$ at $x = 1, 2, 3$, and $p_Y(y) = \sum_{x=1}^{3} \frac{xy}{18} = \frac{y}{3}$ at $x = 1, 2$. Yes X and Y are independent, since the joint frequency function satisfies $p(x, y) = p_X(x)p_Y(y)$ for all $x = 1, 2, 3$ and $y = 1, 2$.

(c) Find $E[1 + X]$, $E[1 + Y]$ and $E[(1 + X)(1 + Y)]$.

$E[X] = \sum_{x=1}^{3} (x)\left(\frac{x}{6}\right) = \frac{7}{3}$; $E[Y] = \sum_{y=1}^{2} (y)\left(\frac{y}{3}\right) = \frac{5}{3}$. Thus, $E[1 + X] = \frac{10}{3}$ and $E[1 + Y] = \frac{8}{3}$

Since X and Y are independent, we obtain $E[(1 + X)(1 + Y)] = E[1 + X] \cdot E[1 + Y] = \frac{80}{9}$.

$$E[g(X,Y)] = \begin{cases} \sum_{x,y} g(x,y)\, p(x,y) \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f(x,y)\,dx\,dy \end{cases}$$

If X and Y are independent

$E[g_1(X) g_2(Y)] = E[g_1(X)]\, E[g_2(Y)]$

Expectation formulas

$\mathrm{Var}(X) = E[X^2] - (E[X])^2$

$\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y]$

$\mathrm{Var}(X \pm Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) \pm 2\,\mathrm{Cov}(X,Y)$

$E[(1+X)(1+Y)] = E[1 + X + Y + XY]$

$= 1 + E[X] + E[Y] + E[XY] \leftarrow E[X]E[Y]$