Q.1 (4 points) Answer the following questions.

(a) Suppose that $X_1, \cdots, X_r$ are independent geometric random variables with parameter p, and that $X = \sum_{k=1}^{r} X_k$. Then find the frequency function $p(i) = P(X = i)$ in terms of r and p.

X is a negative binomial random variable with parameter (r, p). Thus, we obtain

$$p(i) = \binom{i-1}{r-1} p^r (1-p)^{i-r}$$

for $i = r, r+1, \ldots$.

(b) Suppose that X_1 and X_2 are independent Poisson random variables with respective parameters $\lambda_1 = 3$ and $\lambda_2 = 4$. Then find the frequency function $p(i) = P(X = i)$ for $X = X_1 + X_2$.

X is a Poisson random variable with $\lambda = 7$. Thus, we obtain

$$p(i) = e^{-7} \frac{7^i}{i!}$$

for $i = 0, 1, 2, \ldots$.

Q.2 (4 points) The number of cars that a certain dealership sells within a week has a Poisson distribution. The chance of selling no car in a week is exactly $e^{-1.2} \approx 0.3$.

(a) Find the average number of cars sold per week.

$E[X] = 1.2$

(b) Find the probability that one car is sold in a week.

Let X be the number of cars sold in a week. Then we obtain $P(X = 1) = p(1) = e^{-1.2}(1.2) \approx 0.36$

Q.3 (2 points) If a fair coin is tossed repeatedly until it lands heads, compute the conditional probability that a coin is tossed k times given that there are four tails in the first four outcomes.

Let X be the number of times to toss a coin until it lands heads. Then X is a geometric random variable with $p = \frac{1}{2}$. The first four consecutive tails implies that $P(X \geq 5) = \left(\frac{1}{2}\right)^4$. Thus, we can calculate the conditional probability of tossing a coin k times as follows.

$$P(X = k | X \geq 5) = \frac{P(X = k)}{P(X \geq 5)} = \frac{\left(\frac{1}{2}\right)^k}{\left(\frac{1}{2}\right)^4} = \left(\frac{1}{2}\right)^{k-4}$$

for $k = 5, 6, 7, \ldots$.