Q.1 (3 points) Suppose that a random variable X has a normal distribution with $(\mu, \sigma^2) = (1, 16)$.

(a) Find $P(1 < X < 3)$.

$P(1 < X < 3) = P\left(0 < \frac{X-1}{4} < 0.5\right) = \Phi(0.5) - \Phi(0) \approx 0.1915$

(b) Find b so that $P(1 < X < b) = 0.45$.

$P(1 < X < b) = P\left(0 < \frac{X-1}{4} < \frac{b-1}{4}\right) = \Phi\left(\frac{b-1}{4}\right) - \Phi(0) = 0.45$, thus, $\Phi\left(\frac{b-1}{4}\right) = 0.95$. It implies that $\frac{b-1}{4} \approx 1.64$ or 1.65 Therefore, we find $b \approx 7.6$.

Q.2 (3 points) Let X_1 and X_2 be independent standard normal random variables.

(a) Choose the *correct* statement.

(i) $X_1^2 + X_2^2$ has a normal distribution;

(ii) $X_1^2 + X_2^2$ has an exponential distribution.

(ii) $X_1^2 + X_2^2$ is a chi-square with $m = 2$ degrees of freedom, and it is a gamma with $\alpha = 1$ and $\lambda = \frac{1}{2}$. Thus, it has an exponential distribution with $\lambda = \frac{1}{2}$.

(b) Find $P(X_1^2 + X_2^2 \le 1.64)$, and express it in terms of e^M.

$P(X_1^2 + X_2^2 \le 1.64) = 1 - e^{-(1/2)(1.64)} = 1 - e^{-0.82}$

Q.3 (4 points) The chi-square density function with 1 degree of freedom is given by

$$g(x) = \frac{e^{-x/2}x^{-1/2}}{2^{1/2}\Gamma(1/2)} \text{ if } x > 0; \text{ otherwise, } g(x) = 0.$$

Let X be a chi-square random variable with 1 degree of freedom, and let $Y = \sqrt{X}$.

(a) For $t > 0$ find A in $P(Y \le t) = \int_0^A g(x)\,dx$, and express A in terms of t.

Since $P(Y \le t) = P(X \le t^2)$, we must have $A = t^2$

(b) Apply $\Gamma(1/2) = \sqrt{\pi}$, and find the density function f of Y by forming $P(Y \le t) = \int_0^t f(y)\,dy$.

By substituting $x = y^2$ and $dx = 2ydy$ in $\int_0^{t^2} g(x)\,dx$, we obtain

$$P(Y \le t) = \int_0^t \sqrt{\frac{2}{\pi}} e^{-y^2/2}\,dy.$$

Thus, the density function becomes $f(y) = \sqrt{\frac{2}{\pi}} e^{-y^2/2}$, $y > 0$.