Q.1 (4 points) Suppose that n is a positive integer, and that X has the pdf

$$f(x) = \begin{cases} \frac{\lambda^n}{(n-1)!} x^{n-1} e^{-\lambda x} & \text{if } x \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Let $Y = X^n$. Find the pdf $f_Y(y)$ for Y.

Since $y = g(x) = x^n$, we obtain $g^{-1}(y) = y^{1/n}$ and $\frac{d}{dy} g^{-1}(y) = \frac{1}{n} y^{(1-n)/n}$. By the transformation theorem we find

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| = \frac{\lambda^n}{n!} e^{-\lambda y^{1/n}}$$

Q.2 (6 points) Let X and Y be random variables. The joint density function of X and Y is given by

$$f(x, y) = \begin{cases} x + y & \text{if } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1; \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the marginal density $f_X(x)$ of X.

$$f_X(x) = x + \frac{1}{2}, \quad 0 \leq x \leq 1.$$

(b) Find the conditional density $f(y|x)$ of Y given X.

$$f(y|x) = \begin{cases} \dfrac{x + y}{x + 1/2} & \text{if } 0 \leq y \leq 1; \\ 0 & \text{otherwise.} \end{cases}$$

(c) Find the best predictor of Y given X.

$$E(Y|X = x) = \int_{-\infty}^{\infty} y f(y|x) dy = \int_0^1 y \left(\frac{x + y}{x + 1/2} \right) dy = \frac{x/2 + 1/3}{x + 1/2}.$$

Thus, $E(Y|X) = \dfrac{X/2 + 1/3}{X + 1/2}$.