Q.1 (5 points) Suppose that (X, Y) has a bivariate normal distribution with $\mu_x = 2$ and $\mu_y = 0$, and that the covariance matrix Σ for (X, Y) satisfies

$$\Sigma^{-1} = \begin{bmatrix} 1 & -3/4 \\ 0 & 5/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3/4 & 5/2 \end{bmatrix}$$

Answer each of the following questions.

(a) (X, Y) has the joint density function $\frac{5}{4\pi} \exp\left(-\frac{1}{2}Q(x, y)\right)$. Find a, b, and c in the form of $Q(x, y) = (x - a)^2 + [by + c(x - a)]^2$.

We obtain

$$Q(x, y) = [(x-2)\ y] \begin{bmatrix} 1 & -3/4 \\ 0 & 5/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3/4 & 5/2 \end{bmatrix} \begin{bmatrix} (x-2) \\ y \end{bmatrix} = (x-2)^2 + \{(-3/4)(x-2) + (5/2)y\}^2$$

and find $a = 2$, $b = \frac{5}{2}$ and $c = -\frac{3}{4}$.

(b) Find α and β in the conditional density function $f_{Y|X}(y|x) = \frac{1}{(2/5)\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{y-\alpha(x-\beta)}{(2/5)}\right)^2\right]$

By using the result of (a) we obtain

$$f_{Y|X}(y|x) = \frac{5}{2\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{5}{2}y - \frac{3}{4}(x-2)\right)^2\right] = \frac{1}{(2/5)\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{y - \frac{3}{10}(x-2)}{2/5}\right)^2\right]$$

Thus, we find $\alpha = \frac{3}{10}$ and $\beta = 2$.

Remark. It has a normal distribution with $\mu = \frac{3}{10}(x-2)$ and $\sigma^2 = (2/5)^2$.

(c) Find the conditional expectation $E(Y|X)$.

By the remark above we can find $E(Y|X) = \frac{3}{10}(X-2)$.

Q.2 (5 points) Suppose that X and Y are independent normal random variables with parameters $\mu_x = \mu_y = 1$ and $\sigma_x = \sigma_y = 1$. Suppose that U and V are given by

$$U = 2X + Y;$$
$$V = X - Y.$$

Then the conditional density function $f_{V|U}(v|u)$ has a normal distribution with mean $\mu_v + \rho\frac{\sigma_v}{\sigma_u}(u - \mu_u)$ and variance $\sigma_v^2(1-\rho^2)$.

(a) Find the mean parameters μ_u and μ_v for (U, V).

Let $A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$ be the matrix of the linear transformation $\begin{bmatrix} U \\ V \end{bmatrix} = A \begin{bmatrix} X \\ Y \end{bmatrix}$. Then we obtain

$$\begin{bmatrix} \mu_u \\ \mu_v \end{bmatrix} = A \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

Thus, we have $\mu_u = 3$ and $\mu_v = 0$.

(b) Find the parameters σ_u^2, σ_v^2 and ρ for the joint distribution of (U, V).

We can calculate the covariance matrix for (U, V) by

$$\begin{bmatrix} \sigma_u^2 & \rho\sigma_u\sigma_v \\ \rho\sigma_u\sigma_v & \sigma_v^2 \end{bmatrix} = A\Sigma A^T = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix}$$

where Σ is the covariance matrix for (X, Y). Thus, we find $\sigma_u^2 = 5$ and $\sigma_v^2 = 2$. Then the correlation coefficient $\rho = \frac{1}{\sqrt{(5)(2)}} = \frac{1}{\sqrt{10}}$ can be obtained.

(c) Provided $U = t$, find the best predictor of V.

It is the mean value of the conditional density $f_{V|U}(v|t)$, and calculated as

$$E(V|U = t) = \mu_v + \rho\frac{\sigma_v}{\sigma_u}(t - \mu_u) = \frac{t}{5} - \frac{3}{5}$$