Q.1 (2 points) Let $X_1, \ldots, X_9$ be iid normal random variables with mean μ and variance σ^2, and let $\bar{X} = \frac{1}{9}\sum_{i=1}^{9} X_i$ and $S^2 = \frac{1}{8}\sum_{i=1}^{9}(X_i - \bar{X})^2$. Then $a \times \left(\frac{\bar{X} - \mu}{S}\right)$ has a t-distribution by choosing some constant value a. Find the value a, and the specific degrees of freedom for the t-distribution.

Recall "sampling distributions under normal assumption" in lecture note No.6: It has a t-distribution with 8 degrees of freedom when $a = 3$.

Q.2 (4 points) Let X, Y, Z are iid standard normal random variables.

(a) What do you call the exact distribution for $X^2 + Y^2 + Z^2$?

It is a χ^2-distribution with 3 degrees of freedom.

(b) Find the value b so that $P\left(\left|\frac{Z}{\sqrt{X^2 + Y^2}}\right| \le b\right) = 0.95$.

Since $\frac{Z}{\sqrt{(X^2+Y^2)/2}}$ has a t-distribution with 2 degrees of freedom, we obtain

$$P\left(\left|\frac{Z}{\sqrt{(X^2 + Y^2)/2}}\right| \le 4.303\right) = 0.95$$

from $t_{.025} = 4.303$ with $df = 2$. Therefore, $b = 4.303/\sqrt{2}$ (≈ 3.043).

Q.3 (4 points) Suppose that X_1, X_2, X_3, X_4 are iid exponential random variables with parameter $\lambda = 1$.

(a) Find the pdf for $V = \max(\min(X_1, X_2), \min(X_3, X_4))$.

Observe that $Y_1 = \min(X_1, X_2)$ and $Y_2 = \min(X_3, X_4)$ are iid exponential random variables with $\lambda = 2$. Since $V = Y_{(2)}$, we obtain

$$f_V(t) = 4e^{-2t}(1 - e^{-2t}), \quad t > 0.$$

(b) Let $W = \min(X_1, X_2) + \min(X_3, X_4)$. Identify the exact distribution for W.

The addition $W = Y_1 + Y_2$ becomes χ^2-distribution with $\lambda = 2$ and $\alpha = 2$.