Q.1 (4 points) Let $X_1, \ldots, X_n$ be iid exponential random variables with pdf

$$f(x) = (1/\theta)e^{-x/\theta}, \quad x \geq 0,$$

when $\theta > 0$. And define $\bar{X}_n = \dfrac{X_1 + \cdots + X_n}{n}$.

(a) Find $E[\bar{X}_n]$ and $\mathrm{Var}(\bar{X}_n)$.

Since $E[X_i] = \theta$ and $E[X_i] = \theta^2$ for each $i = 1, \ldots, n$, we obtain $E[\bar{X}_n] = \theta$ and $\mathrm{Var}(\bar{X}_n) = \theta^2/n$.

(b) Write "$\bar{X}_n$ converges to θ in probability as $n \to \infty$" in terms of definition, and prove it by applying the Chebyshev's inequality.

$\lim_{n\to\infty} P(|\bar{X}_n - \theta| > \varepsilon) = 0$ for any $\varepsilon > 0$. Since $E[\bar{X}_n] = \theta$, we can show that

$$P(|\bar{X}_n - \theta| > \varepsilon) \leq \frac{\mathrm{Var}(\bar{X}_n)}{\varepsilon^2} = \frac{\theta^2/n}{\varepsilon^2} \to 0$$

as $n \to \infty$.

Q.2 (3 points) Let $X_1, \ldots, X_n$ be iid random variables with $E[X_i] = 2$ and $\mathrm{Var}(X_i) = \sigma^2$, and let $Z_n = \dfrac{\sum_{i=1}^n X_i - 2n}{\sigma\sqrt{n}}$. Apply the central limit theorem and answer the following questions.

(a) What is the exact distribution to which the random variable Z_n converges in distribution?

By the central limit theorem we know that Z_n converges to $N(0, 1)$ in distribution.

(b) What is the exact distribution to which $\dfrac{\sum_{i=1}^n X_i - 2n}{\sqrt{n}}$ converges in distribution in terms of σ?

Since $\sigma Z_n = \dfrac{\sum_{i=1}^n X_i - 2n}{\sqrt{n}}$ and Z_n converges to $N(0, 1)$, the limiting distribution of σZ_n is $N(0, \sigma^2)$.

Q.3 (3 points) Let W be a binomial random variable with parameter $n = 400$ and $p = 0.5$. Explain how to approximate $P(185 \leq W \leq 225)$. Use the standard normal distribution table, and obtain the answer.

W is approximated by a normal random variable Z with mean $\mu = np = 200$ and variance $\sigma^2 = np(1-p) = 100$. Thus, we have

$$P(185 \leq W \leq 225) \approx P(184.5 \leq Z \leq 225.5) = \Phi(2.55) - \Phi(-1.55) \approx 0.934$$