Probabilistic models

Kolmogorov (Andrei Nikolaevich, 1903–1987) put forward an axiomatic system for probability theory. "Foundations of the Calculus of Probabilities," published in 1933, immediately became the definitive formulation of the subject. The fundamental notions for probabilistic models are sample space, events, and probability of an event.

Sample space.

The set of all possible outcomes of an experiment is called the *sample space*, and is typically denoted by Ω. For example, if the outcome of an experiment is the order of finish in a race among 3 boys, Jim, Mike and Tom, then the sample space becomes

$$\Omega = \{(J, M, T), (J, T, M), (M, J, T), (M, T, J), (T, J, M), (T, M, J)\}.$$

In other example, suppose that a researcher is interested in the lifetime of a transistor, and measures it in minutes. Then the sample space is represented by

$$\Omega = \{\text{all nonnegative real numbers}\}.$$

Events.

Any subset of the sample space is called an *event*. In the who-wins-the-race example, "Mike wins the race" is an event, which we denote by A. Then we can write

$$A = \{(M, J, T), (M, T, J)\}.$$

In the transistor example, "the transistor does not last longer than 2500 minutes" is an event, which we denote by E. And we can write

$$E = \{x : 0 \leq x \leq 2500\}.$$

Set operations.

Once we have events $A, B, \ldots$, we can define a new event from these events by using set operations—union, intersection, and complement. The event $A \cup B$, called the *union* of A and B, means that either A or B or both occurs. Consider the "who-wins-the-race" example, and let B denote the event "Jim wins the second place", that is,

$$B = \{(M, J, T), (T, J, M)\}.$$

Then $A \cup B$ means that "either Mike wins the first, or Jim wins the second, or both," that is,

$$A \cup B = \{(M, T, J), (T, J, M), (M, J, T)\}.$$

The event $A \cap B$, called the *intersection* of A and B, means that both A and B occurs. In our example, $A \cap B$ means that "Mike wins the first and Jim wins the second," that is,

$$A \cap B = \{(M, J, T)\}.$$

Set operations, continued.

The event A^c, called the *complement* of A, means that A does not occur. In our example, the event A^c means that "Mike does not win the race", that is,

$$A^c = \{(J, M, T), (J, T, M), (T, J, M), (T, M, J)\}.$$

Now suppose that C is the event "Tom wins the race." Then what is the event $A \cap C$? It is impossible that Mike wins and Tom wins at the same time. In mathematics, it is called the *empty set*, denoted by $\emptyset$, and can be expressed in the form

$$A \cap C = \emptyset.$$

If the two events A and B satisfy $A \cap B = \emptyset$, then they are said to be *disjoint*. For example, "Mike wins the race" and "Tom wins the race" are disjoint events.

Axioms of probability.

A probability P is a "function" defined over all the "events" (that is, all the subsets of Ω), and must satisfy the following properties:

1. If $A \subset \Omega$, then $0 \leq P(A) \leq 1$
2. $P(\Omega) = 1$
3. If $A_1, A_2, \ldots$ are events and $A_i \cap A_j = \emptyset$ for all $i \neq j$ then, $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$.

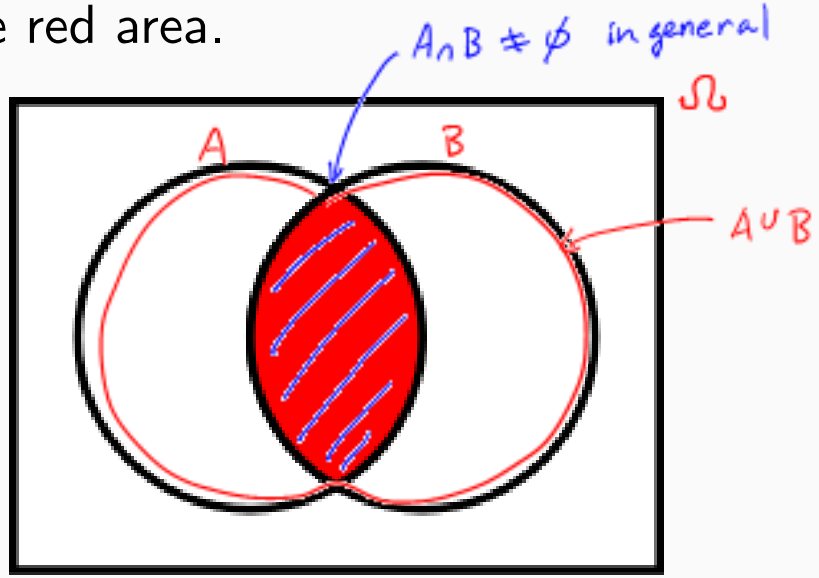
Property 3 is called the "axiom of countable additivity," which is clearly motivated by the property:

$$P(A_1 \cup \cdots \cup A_n) = P(A_1) + \cdots + P(A_n)$$

if $A_1, A_2, \ldots, A_n$ are mutually exclusive events. Quite frankly, there is nothing in one's intuitive notion that requires this axiom for infinite series.

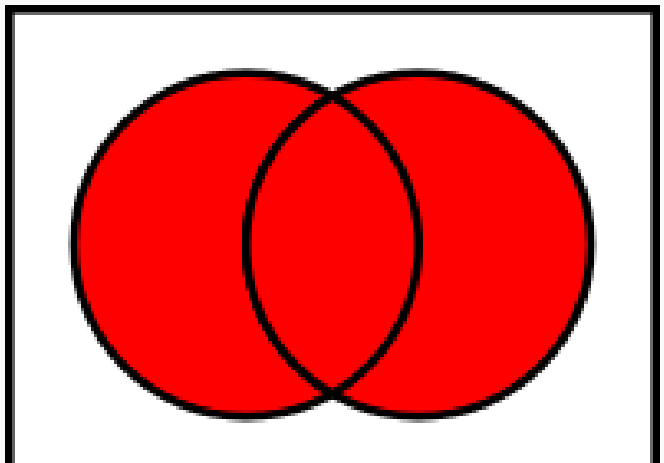
Venn diagram.

The following figure, called **Venn diagram**, represents the probability of $A \cap B$ as shown in the red area.



The next figure now indicates the probability of $A \cup B$ in the red area.

$$P(A \cup B) + P(A \cap B)$$
$$= P(A) + P(B)$$
$$\Downarrow$$
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

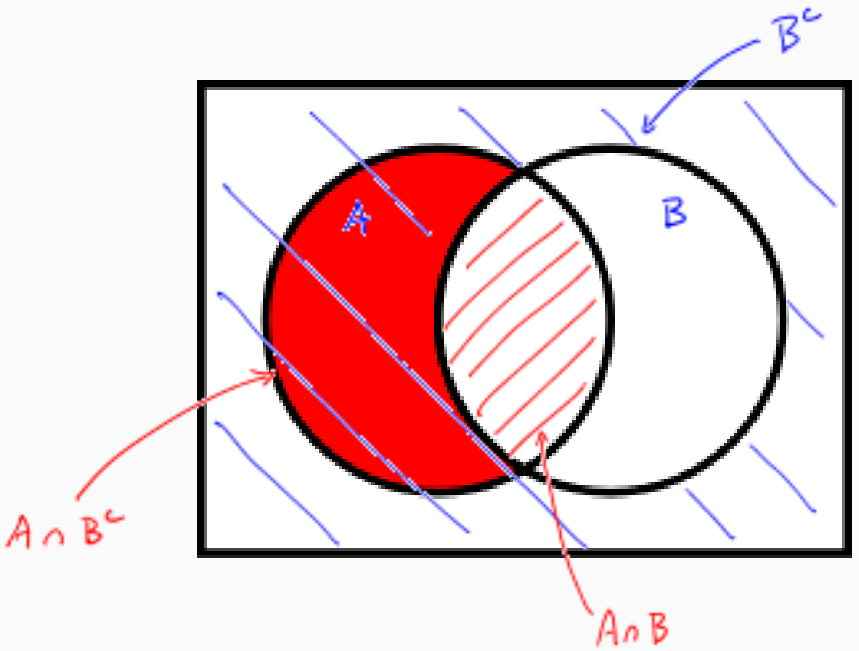


Rules of probability.

A visual illustration of Venn diagram allows us to devise the following addition rule which is not obvious from the axioms of probability.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Another rule $P(A \cap B^c) = P(A) - P(A \cap B)$ can be observed from the following figure.



Sample space having equally likely outcomes.

$1 = P(\{\omega_1, \ldots, \omega_N\}) = P\left(\bigcup_{i=1}^{N} \{\omega_i\}\right) = \sum_{i=1}^{N} P(\{\omega_i\}) = N\, P(\{\omega_i\})$

When the sample space

$$\Omega = \{\omega_1, \ldots, \omega_N\}$$

consists of N outcomes, it is often natural to assume that $P(\{\omega_1\}) = \cdots = P(\{\omega_N\})$. Then we can obtain the probability of a single outcome by

$P(\{\omega_i\}) = P(\{\omega_1\})$

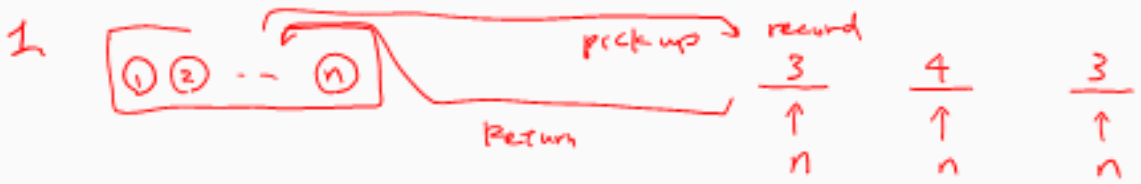
$$P(\{\omega_i\}) = \frac{1}{N} \quad \text{for } i = 1, \ldots, N.$$

Assuming this, we can compute the probability of any event A by counting the number of outcomes in A, and obtain

$$P(A) = \frac{\text{number of outcomes in } A}{N}.$$

$P(\{\omega_1, \omega_2\}) = P(\{\omega_1\}) + P(\{\omega_2\}) = \frac{1}{N} + \frac{1}{N} = \frac{2}{N}$

Combinatorial analysis.



A problem involving equally likely outcomes can be solved by counting. The mathematics of counting is known as **combinatorial analysis.** It can be summarized in three basic methodologies:

1. *Multiplication principle* when individual objects are sampled with replacement and ordered.
2. *Permutations* when individual objects are sampled *without* replacement and ordered.
3. *Combinations* when a subset of individual objects (unordered) are sampled *without* replacement.

Multiplication principle.

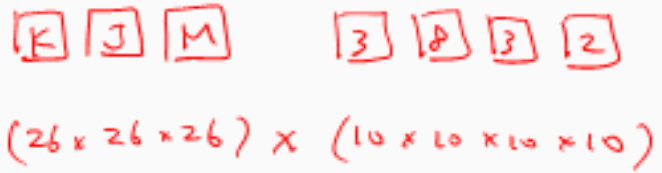
If one experiment has m outcomes, and a second has n outcomes, then there are $m \times n$ outcomes for the two experiments. The **multiplication principle** can be extended and used in the following experiment. Suppose that we have n differently numbered balls in an urn. In the first trial, we pick up a ball from the urn, record its number, and put it back into the urn. In the second trial, we again pick up, record, and put back a ball. Continue the same procedure until the r-th trial. The whole process is called a **sampling with replacement**. By generalizing the multiplication principle, the number of all possible outcomes becomes

$$\underbrace{n \times \cdots \times n}_{r} = n^r.$$

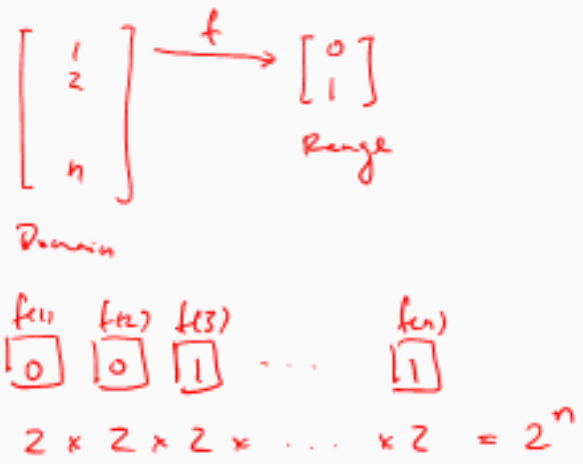
Example

1. How many different 7-place license plates are possible if the first 3 places are to be occupied by letters and the final 4 by numbers?
2. How many different functions f defined on $\{1, \ldots, n\}$ are possible if the value $f(i)$ takes either 0 or 1?

1.



2.

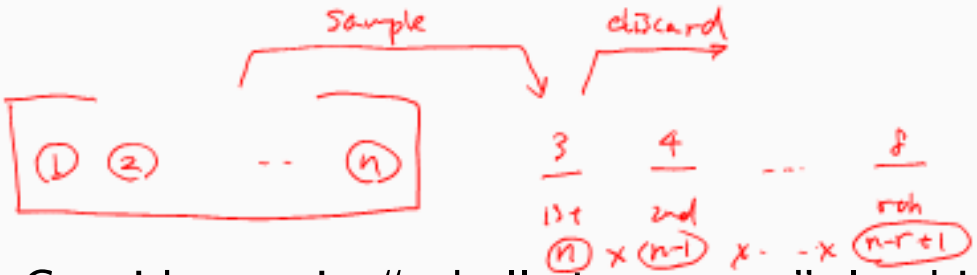


Example

1. How many different 7-place license plates are possible if the first 3 places are to be occupied by letters and the final 4 by numbers?
2. How many different functions f defined on $\{1, \ldots, n\}$ are possible if the value $f(i)$ takes either 0 or 1?

1. $\underbrace{26 \times 26 \times 26}_{\text{3 letters}} \times \underbrace{10 \times 10 \times 10 \times 10}_{\text{4 numbers}} = 175,760,000$
2. n places have either 0 or 1. Thus, we have $\underbrace{2 \times 2 \times \cdots \times 2}_{n} = 2^n$

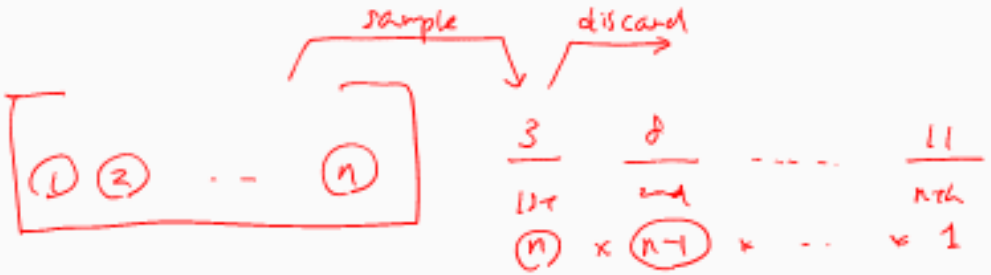
Permutations.



Consider again "n balls in an urn." In this experiment, we pick up a ball from the urn, record its number, but do not put it back into the urn. In the second trial, we again pick up, record, and do not put back a ball. Continue the same procedure r-times. The whole process is called a **sampling without replacement**. Then, the number of all possible outcomes becomes

$$ {}_nP_r = \underbrace{n \times (n-1) \times (n-2) \times \cdots \times (n-r+1)}_{r} $$

Permutations, continued.



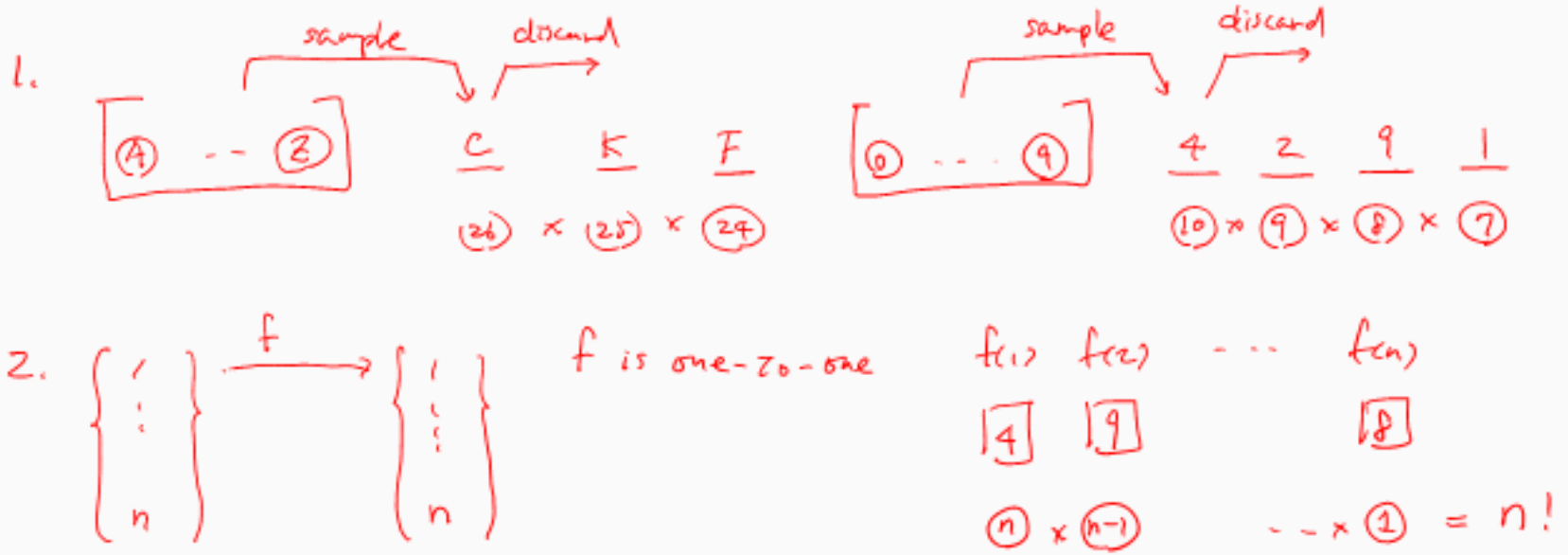
Given a set of n different elements, the number of all the possible "ordered" sets of size r is called r-**element permutation** of an n-element set. In particular, the n-element permutation of an n-element set is expressed as

$$\underbrace{n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1}_{n} = n!$$

The above mathematical symbol "$n!$" is called "n factorial." We define $0! = 1$, since there is one way to order 0 elements.

Example

1. How many different 7-place license plates are possible if the first 3 places are using different letters and the final 4 using different numbers?
2. How many different functions f defined on $\{1, \ldots, n\}$ are possible if the function f takes values on $\{1, \ldots, n\}$ and satisfies that $f(i) \neq f(j)$ for all $i \neq j$?



Example

1. How many different 7-place license plates are possible if the first 3 places are using different letters and the final 4 using different numbers?
2. How many different functions f defined on $\{1, \ldots, n\}$ are possible if the function f takes values on $\{1, \ldots, n\}$ and satisfies that $f(i) \neq f(j)$ for all $i \neq j$?

1. Here different places must have different objects. Thus, we have

$$\underbrace{26 \times 25 \times 24}_{\text{3 different letters}} \times \underbrace{10 \times 9 \times 8 \times 7}_{\text{4 different numbers}} = 78,624,000$$

2. n places must have different numbers from 1 to n. Thus, we have

$$\underbrace{n \times (n-1) \times \cdots \times 2 \times 1}_{\text{n different numbers}} = n!$$

Combinations.

In the experiment, we have n differently numbered balls in an urn, and pick up r balls "as a group." Then there are

$$ {}_nP_r = n \times (n-1) \times \cdots \times (n-r+1) = \frac{n!}{(n-r)!} = \frac{n \times (n-1) \times \cdots \times (n-r+1) \times (n-r) \times \cdots \times 2 \times 1}{(n-r) \times \cdots \times 2 \times 1} $$

ways of selecting the group if the order is relevant (r-element permutation). However, the order is irrelevant when you choose r balls as a group, and any particular r-element group has been counted exactly $r!$ times. Thus, the number of r-element groups can be calculated as

$$ {}_nC_r = \binom{n}{r} = \frac{n!}{(n-r)!\,r!}. $$

permutation

disregard an order.

You should read the above symbol as "n choose r."

How many different ways to order?

In general you $r!$

$3 \times 2 \times 1 = 3!$

Example

1. A committee of 3 is to be formed from a group of 20 people. How many different committees are possible?
2. From a group of 5 women and 7 men, how many different committees consisting of 2 women and 3 men can be formed?

1. 20 chooses 3 $\Rightarrow \binom{20}{3} = \frac{20 \times 19 \times 18}{3 \times 2 \times 1}$

2. 5 chooses 2 $\Rightarrow \binom{5}{2}$

   $\times$

   7 chooses 3 $\Rightarrow \binom{7}{3}$

Example

1. A committee of 3 is to be formed from a group of 20 people. How many different committees are possible?
2. From a group of 5 women and 7 men, how many different committees consisting of 2 women and 3 men can be formed?

1. $\binom{20}{3} = \frac{20 \times 19 \times 18}{3 \times 2 \times 1} = 1140$
2. Here you need to apply the multiplication principle together with combinations.

$$\underbrace{\binom{5}{2}}_{\text{2 women}} \times \underbrace{\binom{7}{3}}_{\text{3 men}} = 350$$

Binomial theorem.

The term "n choose r" is often referred as a binomial coefficient, because of the following identity.

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i\, b^{n-i}.$$

In fact, we can give a proof of the binomial theorem by using combinatorial analysis. For a while we pretend that "commutative law" cannot be used for multiplication. Then, for example, the expansion of $(a+b)^2$ becomes

$(a+b)(a+b) \longrightarrow aa + ab + ba + bb, = a^2 + \binom{2}{1} ab + b^2$

and consists of the 4 terms, aa, ab, ba, and bb. In general, how many terms in the expansion of $(a+b)^n$ should contain a's exactly in i places? The answer to this question indicates the binomial theorem.

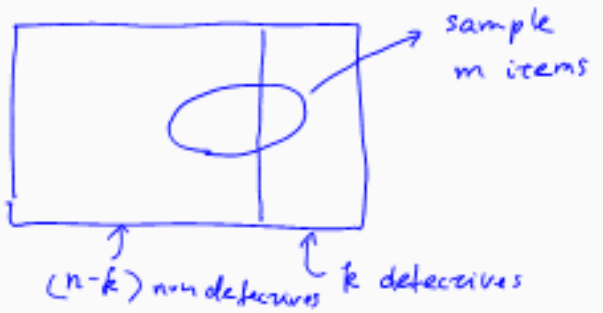
$(a+b)^3 = (a+b)(a+b)(a+b) = aaa + aab + aba + baa + abb + bab + bba + bbb$

$\binom{3}{2} a^2 b \qquad \binom{3}{1} ab^2$

R code: Probabilistic models in quality control.

A lot of n items contains k defectives, and m are selected randomly and inspected. How should the value of m be chosen so that the probability that at least one defective item turns up is 0.9? Apply your answer to the following cases:

1. $n = 1000$ and $k = 10$;
2. $n = 10,000$ and $k = 100$.




```
n = 1000
k = 10
prob <- 1 - dhyper(0, k, n-k, 1:250)
m = match(T, prob >= 0.9)
cat("m =", m, "calculates the probability of", prob[m])
```

i = 0 ; m = 1:250 ⇐ $1 - P(0)$

A = { 0 defectives are found }

A^c = { at least one defective is found }

$P(A^c) = 1 - P(A)$

Example

A committee of 5 students is to be formed from a group of 8 women and 12 men.

1. How many different outcomes can a 5-member committee be formed?
2. How many different outcomes can we form a committee consisting of 2 women and 3 men?
3. What is the probability that a committee consists of 2 women and 3 men?

1. Out of 20 students choose 5 students : $\binom{20}{5}$

2. Out of 8 women choose 2 women : $\binom{8}{2}$

   Out of 12 men choose 3 men : $\binom{12}{3}$

   $\Rightarrow \binom{8}{2} \times \binom{12}{3}$

3. $A = \{$ A committee consists of 2 women and 3 men $\}$

$$P(A) = \frac{\#\text{ of outcomes in } A}{\#\text{ of outcomes in } \mathcal{U}} = \frac{\binom{8}{2}\times\binom{12}{3}}{\binom{20}{5}}$$

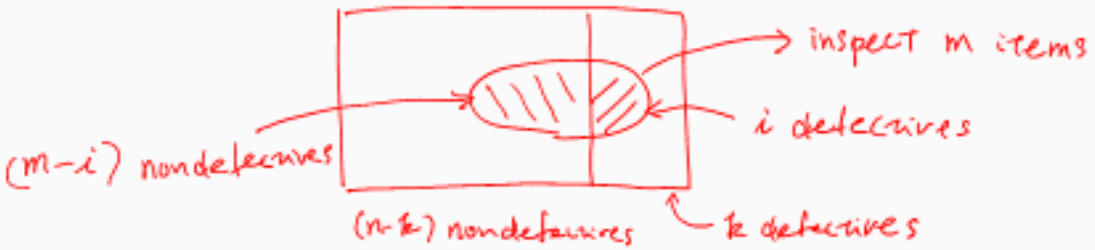
Example

A committee of 5 students is to be formed from a group of 8 women and 12 men.

1. How many different outcomes can a 5-member committee be formed?
2. How many different outcomes can we form a committee consisting of 2 women and 3 men?
3. What is the probability that a committee consists of 2 women and 3 men?

1. $\binom{20}{5} = \frac{20 \times 19 \times 18 \times 17 \times 16}{5!} = 15504$
2. $\binom{8}{2} \times \binom{12}{3} = 6160$
3. $\frac{6160}{15504} \approx 0.40$

Hypergeometric distribution.



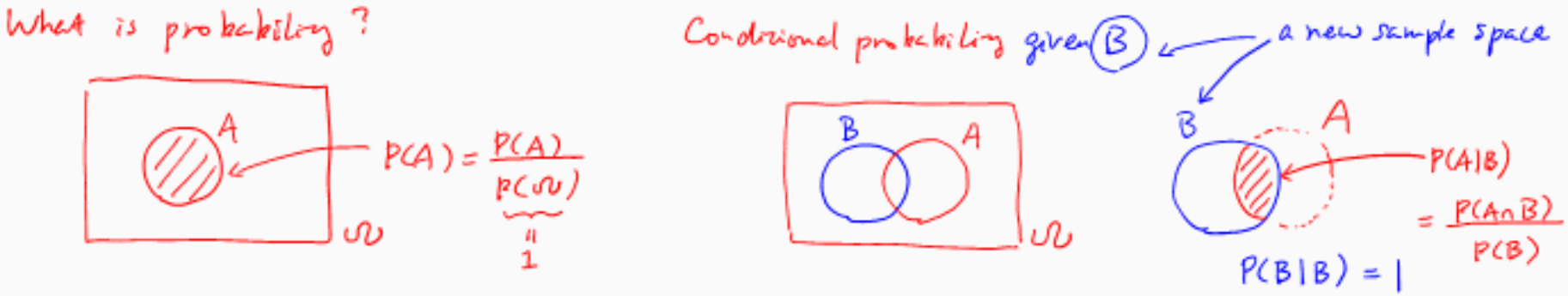
$A = \{$ Exactly i defectives found $\}$

Suppose that we have a lot of size n containing k defectives. If we sample and inspect m random items, what is the probability that we will find i defectives in our sample? This probability is called a *hypergeometric distribution*, and expressed by

$$p(i) = \frac{\binom{k}{i}\binom{n-k}{m-i}}{\binom{n}{m}}, \quad i = \max\{0, m+k-n\}, \ldots, \min\{m, k\}$$

$$P(A) = \frac{\#\text{ of outcomes in } A}{\#\text{ of outcomes in } \Omega} = \frac{\binom{k}{i} \times \binom{n-k}{m-i}}{\binom{n}{m}}$$

Conditional probability.



Let A and B be two events where $P(B) \neq 0$. Then the **conditional probability** of A given B can be defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

The idea of "conditioning" is that "if we have known that B has occurred, the sample space should have become B rather than Ω."

Example

Mr. and Mrs. Jones have two children. What is the conditional probability that their children are both boys, given that they have at least one son?

$\Omega =$ {(B, B), (B, G), (G, B), (G, G)} — (G, B): older, (B, G): younger

$B =$ {(B, B), (B, G), (G, B)}

$A =$ {(B, B)}

$$P(A) = \frac{\#\text{ of outcomes in } A}{\#\text{ of } \text{“ } \text{ in } \Omega} = \frac{1}{4}, \quad P(B) = \frac{3}{4}$$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{1/4}{3/4} = \frac{1}{3}$$

Example

Mr. and Mrs. Jones have two children. What is the conditional probability that their children are both boys, given that they have at least one son?

Let $\Omega = \{(B, B), (B, G), (G, B), (G, G)\}$ be the sample space. Then we can introduce the event $A = \{(B, B)\}$ of both boys, and $B = \{(B, B), (B, G), (G, B)\}$ representing that at least one of them is a boy. Thus, we can find the conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{3/4} = \frac{1}{3}$$

Multiplication rule.

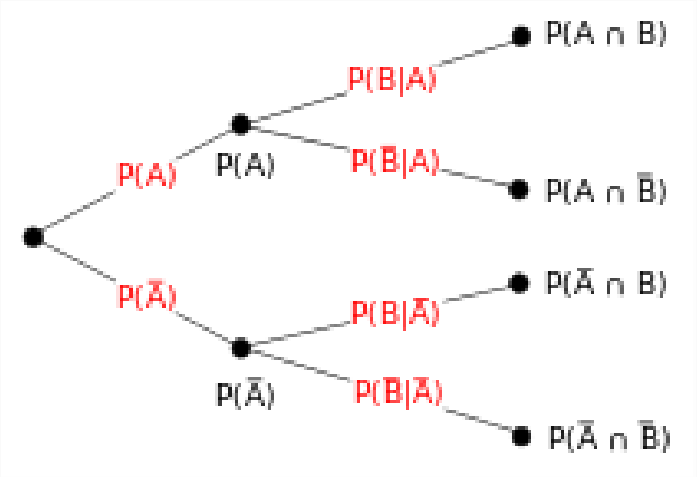
If $P(A)$ and $P(B|A)$ are known then we can use them to find

$P(B|A) = \frac{P(A \cap B)}{P(A)}$ $\Longleftrightarrow$ $P(A \cap B) = P(B|A)P(A)$.

The various forms of multiplication rule are illustrated in the **tree diagram**. The probability that the series of events leading to a particular node will occur is equal to the product of their probabilities.

$A = \{$Take Chem$\}$

$\bar{A} = \{$Take French$\}$



$B = \{$Get an $A\}$

$\bar{B} = \{$Does not get an $A\}$

Example

Celine estimates that her chance to receive an A grade would be $\frac{2}{3}$ in a chemistry course, and $\frac{1}{2}$ in a French course. She decides whether to take a French course or a chemistry course this semester based on the flip of a coin.

1. What is the probability that she gets an A in chemistry?
2. What is the probability that she gets an A in French?

$A = \{\text{Take Chem}\}$, $B = \{\text{Get an A}\}$

$P(B|A) = \frac{2}{3}$ $\quad P(A) = \frac{1}{2}$ $\quad \Longrightarrow P(A \cap B) = P(B|A)\,P(A) = \left(\frac{2}{3}\right)\left(\frac{1}{2}\right) = \frac{1}{3}$

$\bar{A} = \{\text{Take French}\} = A^c$

$P(B|\bar{A}) = \frac{1}{2}$ $\quad P(\bar{A}) = \frac{1}{2}$

$P(\bar{A} \cap B) = P(B|\bar{A})\,P(\bar{A}) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$

Example

Celine estimates that her chance to receive an A grade would be $\frac{2}{3}$ in a chemistry course, and $\frac{1}{2}$ in a French course. She decides whether to take a French course or a chemistry course this semester based on the flip of a coin.

1. What is the probability that she gets an A in chemistry?
2. What is the probability that she gets an A in French?

Let A be the event of choosing the chemistry course, and $\bar{A}$ be the event of choosing the French course. Let B be the event of receiving an A. Then we know that $P(A) = P(\bar{A}) = \frac{1}{2}$ and $P(B|A) = \frac{2}{3}$ and $P(B|\bar{A}) = \frac{1}{2}$

1. $P(A \cap B) = P(B|A)P(A) = \left(\frac{2}{3}\right)\left(\frac{1}{2}\right) = \frac{1}{3}$
2. $P(\bar{A} \cap B) = P(B|\bar{A})P(\bar{A}) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$

Law of total probability.

$$P(B) = P(B \cap A) + P(B \cap A^c)$$

$$P(B|A)P(A) \qquad P(B|A^c)P(A^c)$$

Let A and B be two events. Then we can write the probability $P(B)$ as

$$P(B) = P(B \cap A) + P(B \cap A^c) = P(B|A)P(A) + P(B|A^c)P(A^c).$$

In general, suppose that we have a sequence $A_1, A_2, \ldots, A_n$ of mutually disjoint events satisfying $\bigcup_{i=1}^{n} A_i = \Omega$, where "mutual disjointness" means that $A_i \cap A_j = \emptyset$ for all $i \neq j$. (The events $A_1, A_2, \ldots, A_n$ are called "a partition of Ω.") Then for any event B we have

$$P(B) = \sum_{i=1}^{n} P(B \cap A_i) = \sum_{i=1}^{n} P(B|A_i)P(A_i).$$

$$P(B) = P(B \cap A_1) + \cdots + P(B \cap A_n)$$

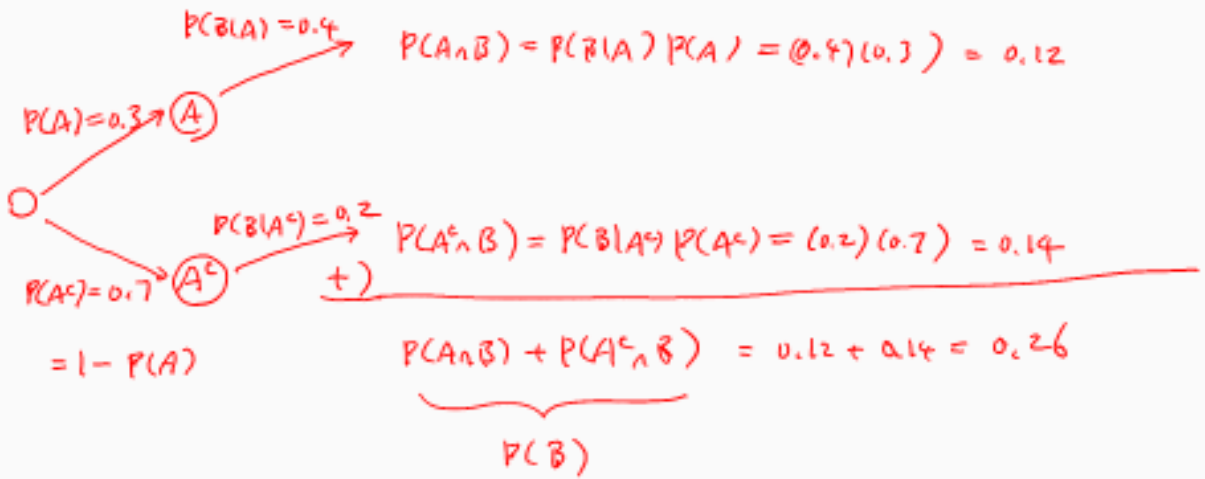
$$P(B|A_1)P(A_1) \qquad P(B|A_n)P(A_n)$$

Example

An study shows that an accident-prone person will have an accident in one year period with probability 0.4, whereas this probability is only 0.2 for a non-accident-prone person. Suppose that we assume that 30 percent of all the new policyholders of an insurance company is accident prone. Then what is the probability that a new policyholder will have an accident within a year of purchasing a policy?

$A = \{$ Accident-prone $\}$ $\quad B = \{$ Have an accident $\}$

$A^c = \{$ non-Accident-prone $\}$



Example

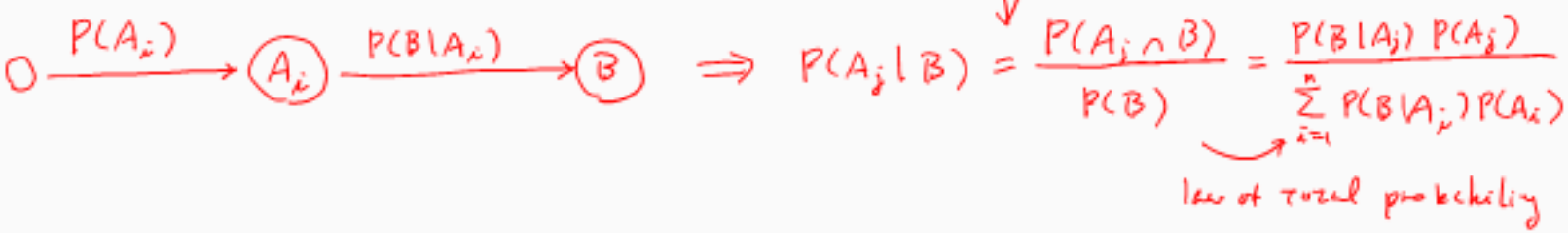
An study shows that an accident-prone person will have an accident in one year period with probability 0.4, whereas this probability is only 0.2 for a non-accident-prone person. Suppose that we assume that 30 percent of all the new policyholders of an insurance company is accident prone. Then what is the probability that a new policyholder will have an accident within a year of purchasing a policy?

Let A be the event that a policy holder is accident-prone, and let B be the event that the policy holder has an accident within a year. Then we know that $P(A) = 0.3$, $P(A^c) = 0.7$, $P(B|A) = 0.4$, and $P(B|A^c) = 0.2$. Hence, we obtain

$$P(B) = P(B|A)P(A) + P(B|A^c)P(A^c) = (0.4)(0.3) + (0.2)(0.7) = 0.26$$

$P(B|A)P(A) = P(A \cap B) = (0.4)(0.3) = 0.12$

Bayes' rule.



Let $A_1, \ldots, A_n$ be events such that A_i's are mutually disjoint, $\bigcup_{i=1}^{n} A_i = \Omega$ and $P(A_i) > 0$ for all i. If we know about the conditional probability $P(B|A_i)$ for every i we can calculate

$$P(A_j|B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)}.$$

Example

Continue from the previous example of insurance business. If a new policy holder had an accident within a year, what is the probability that the policyholder is actually accident-prone?

conditional

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B \mid A)\,P(A)}{P(B)} = \frac{0.12}{0.26}$$

Example

Continue from the previous example of insurance business. If a new policy holder had an accident within a year, what is the probability that the policyholder is actually accident-prone?

Let A be the event that a policy holder is accident-prone, and B be the event that the policy holder has an accident within a year. Then the problem of interest is to find the conditional probability $P(A|B)$.

$P(A) = 0.3 \Rightarrow$ $$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)} = \frac{0.12}{0.26} \approx 0.46$$

Independent events.

$P(A|B) = \frac{P(A \cap B)}{P(B)} = P(A) \iff P(A \cap B) = P(A)P(B) \iff P(B|A) = \frac{P(A \cap B)}{P(A)} = P(B)$

Intuitively we would like to say that A and B are independent if knowing about one event give us no information about another. That is, $P(A|B) = P(A)$ and $P(B|A) = P(B)$. We say A and B are **independent** if

$$P(A \cap B) = P(A)P(B).$$

This definition is symmetric in A and B, and allows $P(A)$ and/or $P(B)$ to be 0. Furthermore, a collection of events $A_1, A_2, \ldots, A_n$ is said to be **mutually independent** if it satisfies

$$P(A_{i_1} \cap \cdots \cap A_{i_m}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_m})$$

for any subcollection $A_{i_1}, A_{i_2}, \ldots, A_{i_m}$.

Example

A coin is tossed three times. Then let A be the event that the first and second tosses match, and let B be the event the second and third tosses match.

1. Define the sample space.
2. Express A, B and $A \cap B$ as subsets of Ω.
3. Are A and B independent? Yes

1. $\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

2. $A = \{HHH, HHT, TTH, TTT\}$ and $B = \{HHH, THH, HTT, TTT\}$

$$P(A) = \frac{\#\text{ of outcomes in } A}{\#\text{ in } \Omega} = \frac{4}{8} = \frac{1}{2}, \quad P(B) = \frac{4}{8} = \frac{1}{2}$$

3. $P(A \cap B) = P(\{HHH, TTT\}) = \frac{2}{8} = \frac{1}{4}$

$P(A)P(B) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$

$\Rightarrow$ Yes.

Remark: If A and B are independent, are A^c and B^c independent? Yes.

Proof: Show $P(A^c \cap B^c) = P(A^c)\,P(B^c)$.

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$P(A^c \cap B^c) = 1 - P(A \cup B) = 1 - [P(A) + P(B) - P(A \cap B)]$$

$$= 1 - [P(A) + P(B) - P(A)P(B)]$$

$$= 1 - P(A) - P(B) + P(A)P(B)$$

$$= 1 - P(A) - P(B)[1 - P(A)]$$

$$= [1 - P(A)][1 - P(B)]$$

$$= P(A^c)\,P(B^c) \quad \text{QED.}$$

$(A^c \cap B^c)^c = A \cup B$

De Morgan's laws

$(A \cap B)^c = A^c \cup B^c$

$(A \cup B)^c = A^c \cap B^c$

Question: Suppose that A and B are independent.

1. Are A and B^c independent? Yes
2. Are A^c and B independent? Yes

Example

A coin is tossed three times. Then let A be the event that the first and second tosses match, and let B be the event the second and third tosses match.

1. Define the sample space.
2. Express A, B and $A \cap B$ as subsets of Ω.
3. Are A and B independent?

1. $\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$,
2. $A = \{HHH, HHT, TTT, TTH\}$, $B = \{HHH, THH, TTT, HTT\}$, and $A \cap B = \{HHH, TTT\}$.
3. $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{2}$, and $P(A \cap B) = \frac{1}{4}$. Thus, A and B satisfy $P(A \cap B) = P(A)P(B)$, and therefore, they are independent.

Homework

Problem

The world series in baseball continues until either the american league (A) or the national league (N) wins four games. How many different outcomes are possible if the series goes

1. *four games?*
2. *five games?*
3. *six games?*
4. *seven games?*

For example, ANNAAA means that the american league wins in six games.

Problem

Two six-sided dice are thrown sequentially, and the face values that come up are recorded.

1. *List the sample space* Ω.
2. *List the elements that make up the following events:*
   - 2.1 $A = \{$*the sum of the two values is at least* $8\}$,
   - 2.2 $B =$ $\{$*the value of the first die is higher than the value of the second*$\}$,
   - 2.3 $C = \{$*the first value is* $4\}$.
3. *Assuming equally likely outcomes, find* $P(A)$, $P(B)$, *and* $P(C)$.
4. *List the events of the following events:*
   - 4.1 $A \cap C$,
   - 4.2 $B \cup C$,
   - 4.3 $A \cap (B \cup C)$.
5. *Again assuming equally likely outcomes, find* $P(A \cap C)$, $P(B \cup C)$, *and* $P(A \cap (B \cup C))$.

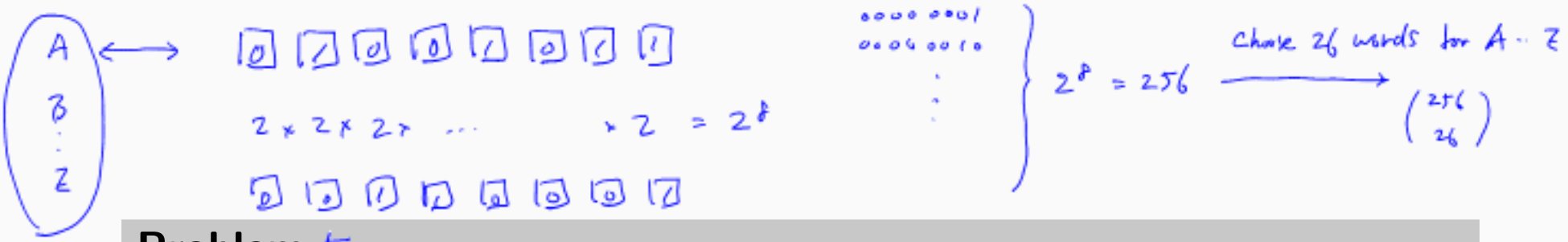
Problem 3

Suppose that $P(A) = 0.4$, $P(B) = 0.5$, *and* $P(A \cap B) = 0.3$. *Then find the following probability:*

1. $P(A \cup B)$
2. $P(A \cap B^c)$
3. $P(A^c \cup B^c)$

Problem

A deck of 52 *cards is shuffled thoroughly. What is the probability that the four aces are all next to each other? (Hint: Imagine that you have* 52 *slots lined up to place the four aces. How many different ways can you "choose" four slots for those aces? How many different ways do you get consecutive slots for those aces?)*



Problem 5

Express the answer by combinations in each of the following questions:

1. *How many ways are there to encode the 26-letter English alphabet into 8-bit binary words (sequences of eight 0's and 1's)?*
2. *What is the coefficient of x^3y^4 in the expansion of $(x+y)^7$?*
3. *A child has six blocks, three of which are red and three of which are green. How many patterns can she make by placing them all in a line?*

Problem

From a group of 5 students, Amanda, Brad, Carey, David and Eric, we want to form a committee consisting of 3 students.

1. *How many different ways to choose committee members?*
2. *Now suppose that Amanda and Brad refuse to serve together. Then how many different ways to choose committee members?*

Problem

A couple has two children.

1. *What is the probability that both are girls given that the oldest is a girl?*
2. *What is the probability that both are girls given that one of them is a girl?*

Problem

Urn 1 has three red balls and two white balls, and urn 2 has two red balls and five white balls. A fair coin is tossed; if it lands heads up, a ball is drawn from urn 1, and otherwise, a ball is drawn from urn 2.

1. *What is the probability that a red ball is drawn?*
2. *If a red ball is drawn, what is the probability that the coin landed heads up?*

Problem 9

An urn contains three red and two white balls. A ball is drawn, and then it and another ball of the same color are placed back in the urn. Finally, a second ball is drawn.

1. *What is the probability that the second ball drawn is white?*
2. *If the second ball drawn is white, what is the probability that the first ball drawn was red?*

Problem 10

Let A and B be independent events with $P(A) = 0.7$ and $P(B) = 0.2$. Compute the following probabilities:

1. $P(A \cap B)$
2. $P(A \cup B)$
3. $P(A^c \cup B^c)$

Bet on numbers: 3 6 4 4 1 6

1 2 3 4 5 6

match ⇒ You win

Problem

A die is rolled six times. If the face numbered j is the outcome on the j-th roll, we say "a match has occurred." You win if at least one match occurs during the six trials.

1. *Let A_i be the event that i is observed on the i-th roll. Find $P(A_i)$ and $P(A_i^c)$ for each $i = 1, 2, 3, 4, 5, 6$.* $P(A_i) = \frac{1}{6}$, $P(A_i^c) = 1 - P(A_i) = \frac{5}{6}$
2. *Find $P(A_1^c \cap A_2^c \cap A_3^c \cap A_4^c \cap A_5^c \cap A_6^c)$*
3. *Let B be the event that at least one match occurs. Express B^c in terms of A_1, A_2, A_3, A_4, A_5, and A_6.*
4. *Find the probability that you win.*

2. $P(A_1^c \cap \cdots \cap A_6^c) = P(A_1^c) \cdots P(A_6^c) = \left(\frac{5}{6}\right)^6$

3. $B^c = A_1^c \cap \cdots \cap A_6^c$

4. $P(B) = 1 - P(B^c) = 1 - \left(\frac{5}{6}\right)^6 \approx 0.665$

Optional problem

Optional problem.

Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a prize; behind the others, goats. You pick a door, say No. 1, and the show's host, Monty Hall, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Now you will respond to him according to one of the following strategies:

I. Stick with the one you have chosen. Thus, say "no" to Monty.

II. Switch it. Thus, say "yes" to Monty.

III. Flip a coin to decide it. Thus, say "yes" with probability $1/2$ (and "no" with probability $1/2$).

Optional problem, continued.

Calculate the actual probability of winning for each strategy by completing the following questions.

1. Let A be the event that you pick the door with prize at the first time. Find $P(A)$ and $P(A^c)$
2. Let B be the event that you choose the door with prize at the second time after Monty shows another door with a goat. Determine $P(B|A)$ and $P(B|A^c)$ for each strategy, I, II, and III. Hint: $P(B|A) = 1$ and $P(B|A^c) = 0$ for strategy I.
3. Calculate $P(B)$ for each strategy.

Answers to homework

Problem 1.

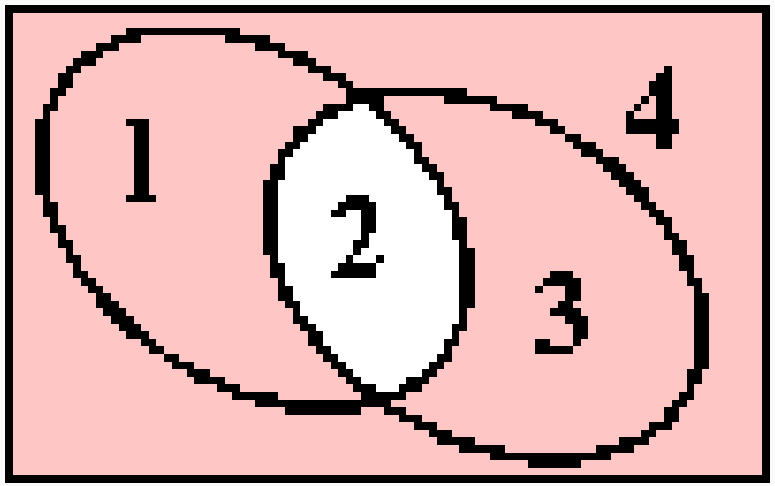
1. The series goes only four games only when AAAA or NNNN. Thus, only two outcomes.
2. The series goes five games if $\square\square\square\square A$ with only one N in $\square$'s, or $\square\square\square\square N$ with only one A in $\square$'s. Thus, we have $4 + 4 = 8$ outcomes.
3. The series goes six games if $\square\square\square\square\square A$ with exactly two N's in $\square$'s, or $\square\square\square\square\square N$ with exactly two A's in $\square$'s. Thus, we have $\binom{5}{2} + \binom{5}{2} = 20$ outcomes.
4. By now you must get the idea. The answer is $\binom{6}{3} + \binom{6}{3} = 40$ outcomes.

Problem 2.

1. $\Omega = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),$
   $(2,2),\ldots,(6,6)\}$.
   Note that $(1,2)$ and $(2,1)$ are distinct outcomes, and that the part "..." should be obvious to you.
2. 2.1 $A = \{(2,6),(3,5),(3,6),(4,4),(4,5),(4,6),(5,3),\cdots,(5,6),$
   $(6,2),\ldots,(6,6)\}$
   2.2 $B = \{(2,1),(3,1),(3,2),(4,1),\ldots,(4,3),(5,1),\ldots,(5,4),$
   $(6,1),\ldots,(6,5)\}$
   2.3 $C = \{(4,1),\ldots,(4,6)\}$
3. $P(A) = \frac{15}{36} = \frac{5}{12}$, $P(B) = \frac{15}{36} = \frac{5}{12}$, and $P(C) = \frac{6}{36} = \frac{1}{6}$.
4. 4.1 $A \cap C = \{(4,4),(4,5),(4,6)\}$
   4.2 $B \cup C = \{(2,1),(3,1),(3,2),(4,1),\ldots,(4,6),(5,1),\ldots,(5,4),$
   $(6,1),\ldots,(6,6)\}$
   4.3 $A \cap (B \cup C) = \{(4,4),(4,5),(4,6),(5,3),(5,4),(6,2),\ldots,(6,5)\}$
5. $P(A \cap C) = \frac{3}{36} = \frac{1}{12}$, $P(B \cup C) = \frac{18}{36} = \frac{1}{2}$, and
   $P(A \cap (B \cup C)) = \frac{9}{36} = \frac{1}{4}$.

Problem 3.

1. $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.6$
2. $P(A \cap B^c) = P(A) - P(A \cap B) = 0.1$
3. Draw the Venn diagram for the probability of $A^c \cup B^c$, and find the following rule.



$$P(A^c \cup B^c) = 1 - P(A \cap B) = 0.7$$

Problem 4.

Let Ω be the sample space of different outcomes in which four places are chosen for aces out of 52 places, and let A be the event that four places are located next to each other. Then the number of outcomes in Ω is $\binom{52}{4} = 270,725$, and the number of outcomes in A is just 49. Thus, the probability is calculated as $\frac{49}{270725} \approx 0.0002$ (which is very small).

Problem 5.

1. $\binom{2^8}{26} = \binom{256}{26}$
2. $\binom{7}{3}$
3. $\binom{6}{3}$

Problem 6.

1. $\binom{5}{3} = 10$
2. There are three possible outcomes in which Amanda and Brad serve together. Thus, by removing these three cases, we have $10 - 3 = 7$ different ways to form a committee.

Problem 7.

Let $\Omega = \{(B, B), (B, G), (G, B), (G, G)\}$ be the sample space.

1. Here we can introduce the event $A = \{(G, G)\}$ of both girls, and $B = \{(G, G), (B, G)\}$ representing that the older is a girl. Thus, we obtain
$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{1/2} = \frac{1}{2}$$
2. Here we can introduce the event $A = \{(G, G)\}$ of both girls, and $C = \{(G, G), (B, G), (G, B)\}$ representing that at least one of them is a girl. Then we obtain
$$P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{1/4}{3/4} = \frac{1}{3}$$

Problem 8.

Let A be the event that a coin lands head up, and let B be the event that a red ball is drawn. Then we know that $P(A) = P(A^c) = \frac{1}{2}$, $P(B|A) = \frac{3}{5}$, and $P(B|A^c) = \frac{2}{7}$.

1. We can apply the law of total probability, and calculate $P(B)$ by

$$P(B|A)P(A) + P(B|A^c)P(A^c) = \left(\frac{3}{5}\right)\left(\frac{1}{2}\right) + \left(\frac{2}{7}\right)\left(\frac{1}{2}\right) = \frac{31}{70}$$

2. We now use the definition of conditional probability, and obtain

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)} = \frac{(3/5)(1/2)}{(31/70)} = \frac{21}{31}$$

Problem 9.

Let A be the event that the first ball is red, and let B be the event that the second ball is white. Then we know that $P(A) = \frac{3}{5}$, $P(A^c) = \frac{2}{5}$, $P(B|A) = \frac{2}{6} = \frac{1}{3}$, and $P(B|A^c) = \frac{3}{6} = \frac{1}{2}$.

1. We can apply the law of total probability, and calculate $P(B)$ by

$$P(B|A)P(A) + P(B|A^c)P(A^c) = \left(\frac{1}{3}\right)\left(\frac{3}{5}\right) + \left(\frac{1}{2}\right)\left(\frac{2}{5}\right) = \frac{2}{5}$$

2. We now use the definition of conditional probability, and obtain

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)} = \frac{(1/3)(3/5)}{(2/5)} = \frac{1}{2}$$

Problem 10.

1. $P(A \cap B) = P(A)P(B) = (0.7)(0.2) = 0.14$
2. $P(A \cup B) = P(A) + P(B) - P(A \cap B) = (0.7) + (0.2) - (0.14) = 0.76$
3. We must use the fact that A^c and B^c are independent, and obtain

$$\begin{aligned} P(A^c \cup B^c) &= P(A^c) + P(B^c) - P(A^c \cap B^c) \\ &= P(A^c) + P(B^c) - P(A^c)P(B^c) \\ &= (0.3) + (0.8) - (0.3)(0.8) = 0.86 \end{aligned}$$

Problem 11.

1. $P(A_i) = \frac{1}{6}$ and $P(A_i^c) = 1 - P(A_i) = \frac{5}{6}$ for each $i = 1, 2, 3, 4, 5, 6$.
2. Since they are independent, we have

$$P(A_1^c \cap A_2^c \cap A_3^c \cap A_4^c \cap A_5^c \cap A_6^c)$$

$$= P(A_1^c)P(A_2^c)P(A_3^c)P(A_4^c)P(A_5^c)P(A_6^c) = \left(\frac{5}{6}\right)^6$$

3. The complement B^c represents the event that no matches occur. Thus, we have

$$B^c = A_1^c \cap A_2^c \cap A_3^c \cap A_4^c \cap A_5^c \cap A_6^c$$

4. $P(B) = 1 - P(B^c) = 1 - \left(\frac{5}{6}\right)^6$