Q.1 (20 points) Bowl I contains 3 red chips and 3 blue chips. Then answer the following questions.

(a) *Two chips* are drawn from *bowl I* at random without replacement. Let A be the event that at least one chip is blue from bowl I. Find $P(A)$.

A^c becomes the event that two red chips are drawn. $P(A) = 1 - P(A^c) = 1 - \dfrac{\binom{3}{2}}{\binom{6}{2}} = \frac{4}{5}$

(b) Suppose that *four* of these 6 chips are selected at random without replacement, and put in *bowl II*. Let B be the event that 2 red chips and 2 blue chips are transferred to bowl II. Find $P(B)$.

$$P(B) = \frac{\binom{3}{2} \times \binom{3}{2}}{\binom{6}{4}} = \frac{3}{5},$$

(c) Now *two chips* are drawn from *bowl II* at random without replacement. Let C be the event that at least one chip is blue from bowl II. Find the conditional probability $P(C|B)$.

$$P(C|B) = 1 - \frac{\binom{2}{2}}{\binom{4}{2}} = \frac{5}{6}.$$

(d) Given that *at least one blue chip* is drawn from *bowl II*, find the *conditional probability* that *bowl II* consists of 2 red chips and 2 blue chips, that is, find $P(B|C)$. Hint: Since we randomly choose two chips at the end in either of the events, A or C, $P(A)$ and $P(C)$ are the same.

$$P(B|C) = \frac{P(C|B)P(B)}{P(C)} = \frac{5}{8}.$$

Q.2 (20 points) If a drawn card is numbered j on the j-th trial, you win. At each trial the previously drawn card is returned, the deck of 52 cards is thoroughly shuffled, and a new card is drawn. You win the game if at least "one win" occurs during 13 trials.

(a) Let A_i be the event that a card numbered i is drawn on the i-th attempt. Find $P(A_i)$ and $P(A_i^c)$ for each $i = 1, 2, \ldots, 13$.

$P(A_i) = \frac{4}{52} = \frac{1}{13}$ and $P(A_i^c) = 1 - P(A_i) = \frac{12}{13}$ for each $i = 1, 2, \ldots, 13$.

(b) Let B be the event that you never win in all the 13 trials. Derive a formula for $P(B)$.

Observe that
$$B = A_1^c \cap A_2^c \cap \cdots \cap A_{13}^c$$

Since they are independent, we have

$$P(A_1^c \cap A_2^c \cap \cdots \cap A_{13}^c) = P(A_1^c) \times P(A_2^c) \times \cdots \times P(A_6^c) = \left(\frac{12}{13}\right)^{13}$$

Thus, we obtain $P(B) = \left(\frac{12}{13}\right)^{13}$.

(c) Derive a formula for the probability that you win the game.

$P(B^c) = 1 - P(B) = 1 - \left(\frac{12}{13}\right)^{13}$

Q.3 (20 points) Let X and Y be two random variables. We know that $E[X] = \frac{1}{2}$, $E[X^2] = \frac{7}{4}$, $E[Y] = 1$, $E[Y^2] = 2$, and $E[XY] = \frac{3}{4}$.

(a) Find $E[X + Y]$.

$E[X + Y] = E[X] + E[Y] = \frac{3}{2}$

(b) Find $\mathrm{Var}(X)$.

$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = \frac{3}{2}$

(c) Find $\mathrm{Cov}(X, Y)$.

$\mathrm{Cov}(X, Y) = E[XY] - E[X]E[Y] = \frac{1}{4}$.

(d) Find $\mathrm{Var}(X + Y)$.

$\mathrm{Var}(Y) = E[Y^2] - (E[Y])^2 = 1$. Thus, $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + 2\mathrm{Cov}(X, Y) + \mathrm{Var}(Y) = 3$.

Q.4 (20 points) A pair (X, Y) of discrete random variables has a joint frequency function

$$p(x, y) = \frac{x + y}{12}, \quad x = 1, 2 \text{ and } y = 1, 2.$$

(a) Find $P(X + Y = 3)$.

$P(X + Y = 3) = p(1, 2) + p(2, 1) = 1/2$.

(b) Find the marginal frequency functions of X and Y.

$$p_X(x) = \sum_{y=1}^{2} \frac{x + y}{12} = \frac{2x + 3}{12} \text{ at } x = 1, 2, \text{ and } p_Y(y) = \sum_{x=1}^{2} \frac{x + y}{12} = \frac{2y + 3}{12} \text{ at } y = 1, 2.$$

(c) Find $E[X]$ and $E[Y]$.

$$E[X] = \sum_{x=1}^{2} (x) \left(\frac{2x+3}{12}\right) = \frac{19}{12}; \ E[Y] = \sum_{y=1}^{2} (y) \left(\frac{2y+3}{12}\right) = \frac{19}{12}.$$

(d) Find $E[XY]$.

$$E[XY] = \sum_{x=1}^{2} \sum_{y=1}^{2} (xy) \left(\frac{x+y}{12}\right) = \frac{5}{2}.$$

(e) Are X and Y independent? Justify your answer.

No, since $E[XY] \neq E[X]E[Y]$.

Q.5 (20 points) Let X and Y be continuous random variables. The joint density function of X and Y is given by

$$f(x, y) = \begin{cases} \frac{xy}{4} & \text{if } 0 \leq x \leq 2 \text{ and } 0 \leq y \leq 2; \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $P(X \leq 1, Y \leq 1)$.

$$P(X \leq 1, Y \leq 1) = \int_0^1 \int_0^1 \frac{xy}{4} \, dx \, dy = \frac{1}{16}.$$

(b) Find the marginal densities of X and Y.

$$f_X(x) = \int_0^2 \frac{xy}{4} \, dy = \frac{x}{2}, \ 0 \leq x \leq 2;$$

$$f_Y(y) = \int_0^2 \frac{xy}{4} \, dx = \frac{y}{2}, \ 0 \leq y \leq 2.$$

(c) Are X and Y independent?

Yes, since the joint density function satisfies $f(x,y) = f_X(x)f_Y(y)$.

(d) Let n be a positive integer. Calculate $E[X^n]$ and $E[Y^n]$ in terms of n.

$$\int_0^2 x^n f_X(x) \, dx = \int_0^2 \frac{x^{n+1}}{2} \, dx = \frac{2^{n+1}}{n+2}$$

$$\int_0^2 y^n f_Y(y) \, dy = \int_0^2 \frac{y^{n+1}}{2} \, dy = \frac{2^{n+1}}{n+2}$$

(e) Find $E[(XY)^n]$ in terms of n.

Since X and Y independent, we have $E[(XY)^n] = E[X^n]E[Y^n] = \frac{2^{2n+2}}{(n+2)^2}$.