Q.1 (20 points) Let X and Y be random variables. The joint density function of X and Y is given by

$$f(x,y) = \begin{cases} 8xy & \text{if } 0 \le x \le y \le 1; \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the marginal density $f_X(x)$ of X.

$$f_X(x) = \int_x^1 8xy\,dy = 4x(1-x^2), \quad 0 \le x \le 1.$$

(b) Find the conditional density $f(y|x)$ of Y given $X = x$.

$$f(y|x) = \begin{cases} \dfrac{2y}{1-x^2} & \text{if } x \le y \le 1; \\ 0 & \text{otherwise.} \end{cases}$$

(c) Find the best predictor of Y given X.

$$E(Y|X=x) = \int_{-\infty}^{\infty} yf(y|x)dy = \int_x^1 y\left(\frac{2y}{1-x^2}\right)dy = \frac{2(1-x^3)}{3(1-x^2)} = \frac{2(1+x+x^2)}{3(1+x)}$$

Thus, $E(Y|X) = \dfrac{2(1-X^3)}{3(1-X^2)} = \dfrac{2(1+X+X^2)}{3(1+X)}$.

Q.2 (20 points) The function $f(x,y) = \frac{1}{16\pi}\exp\left(-\frac{1}{2}Q(x,y)\right)$ with the quadratic form

$$Q(x,y) = (x-7)^2 + \left(\frac{y-6x-58}{8}\right)^2$$

gives the joint density function for (X,Y).

(a) Find the marginal density function $f_X(x)$ for X.
$f_X(x) = \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}(x-7)^2\right]$

(b) Find the conditional density function $f_{Y|X}(y|x)$.
$f_{Y|X}(y|x) = \frac{1}{8\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{y-6x-58}{8}\right)^2\right\}$

(c) Find the conditional expectation $E(Y|X)$ and the conditional variance $\text{Var}(Y|X)$.
$E(Y|X) = 6X + 58$ and $\text{Var}(Y|X) = (8)^2$

Q.3 (20 points) Suppose that X_1 and X_2 are independent normal random variables with parameters (μ_1, σ_1^2), (μ_2, σ_2^2), and that $\mu_1 = 3$ and $\mu_2 = 1$. Consider the linear transformation $U = g_1(X_1, X_2) = X_1 + X_2 + 5$ and $V = g_2(X_1, X_2) = 2X_2 + 9$. Then we know that (U, V) has a bivariate normal distribution with parameters μ_u, μ_v, σ_u, σ_v, and ρ.

(a) We can express $\begin{bmatrix} U \\ V \end{bmatrix} = A \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \mathbf{b}$. Find the matrix A and the vector $\mathbf{b}$.

$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$

(b) Find the mean parameters μ_u and μ_v.

$$\begin{bmatrix} \mu_u \\ \mu_v \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix} + \mathbf{b} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ 9 \end{bmatrix} = \begin{bmatrix} 9 \\ 11 \end{bmatrix}$$

(c) Find σ_u, σ_v, and ρ in terms of σ_1 and σ_2.

The covariance matrix for (U, V) is given by

$$A\Sigma A^T = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \sigma_1^2 + \sigma_2^2 & 2\sigma_2^2 \\ 2\sigma_2^2 & 4\sigma_2^2 \end{bmatrix}$$

Thus, $\sigma_u = \sqrt{\sigma_1^2 + \sigma_2^2}$, $\sigma_v = 2\sigma_2$ and $\rho = \dfrac{\sigma_2}{\sqrt{\sigma_1^2 + \sigma_2^2}}$.

Remark. The direct calculation of $\text{Cov}(U, V)$ can be obtained as follows:

$$\begin{aligned} \text{Cov}(U, V) &= E[(U - \mu_u)(V - \mu_v)] = E[(X_1 + X_2 - 4)(2X_2 - 2)] \\ &= 2E[(X_1 - \mu_1)(X_2 - \mu_2)] + 2E[(X_2 - \mu_2)^2] = 2\text{Var}(X_2) = 2\sigma_2^2 \end{aligned}$$

$\text{Var}(X)$ and $\text{Var}(Y)$ can be similarly calculated.

Q.4 (20 points) Let

$$f(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1 - x)^{\beta-1}, \quad 0 < x < 1,$$

be a beta distribution with parameter (α, β). Answer the following questions.

(a) Suppose that $X_1, \ldots, X_n$ are iid random variables from the uniform distribution having the pdf $f(x) = 1$, $0 < x < 1$, and that $X_{(k)}$ is the k-th order statistic. Find $X_{(k)}$ having a beta distribution, and identify α and β in terms of n and k.

The k-th order statistic $X_{(k)}$ has the pdf $f(x) = \frac{n!}{(k-1)!(n-k)!} x^{k-1}(1 - x)^{n-k}$. Thus, $\alpha = k$ and $\beta = n - k + 1$.

(b) Find $E[X_{(k)}^3]$ in terms of n and k.

$$\begin{aligned} E[X_{(k)}^3] &= \int_0^1 x^3 f(x)\, dx = \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n-k+1)} \int_0^1 x^{k+2}(1 - x)^{n-k}\, dx \\ &= \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n-k+1)} \left(\frac{\Gamma(k+3)\Gamma(n-k+1)}{\Gamma(n+4)} \right) = \frac{(k+2)(k+1)k}{(n+3)(n+2)(n+1)} \end{aligned}$$

Q.5 (20 points) Let $X_1, \ldots, X_n$ be iid normal random variables with mean μ and variance σ^2, and let $\bar{X} = \dfrac{1}{n} \displaystyle\sum_{i=1}^{n} X_i$ and $S^2 = \dfrac{1}{n-1} \displaystyle\sum_{i=1}^{n} (X_i - \bar{X})^2$.

(a) $k \times \left(\dfrac{S^2}{\sigma^2}\right)$ is related to some type of distribution by multiplying it by some constant value k. Give the name of distribution with specific degrees of freedom, and find k in terms of n.

It has a χ^2-distribution with $n-1$ degrees of freedom when $k = n-1$.

(b) $k \times \left(\dfrac{\bar{X}-\mu}{S}\right)$ is related to some type of distribution by multiplying it by some constant value k. Give the name of distribution with specific degrees of freedom, and find k in terms of n.

It has a t-distribution with $n-1$ degrees of freedom when $k = \sqrt{n}$.

(c) Suppose that $n = 9$. Then find the value k so that $P\left(\dfrac{\bar{X}-\mu}{S} < k\right) = 0.05$.

$$k = -\frac{t_{0.05,8}}{\sqrt{9}} = -1.86/3 = -0.62.$$

(d) Suppose that $n = 25$. Then find the value k so that $P\left(\left|\dfrac{\bar{X}-\mu}{S}\right| \leq k\right) = 0.95$.

$$k = \frac{t_{0.025,24}}{\sqrt{25}} = 2.064/5 = 0.4128.$$