Q.1 (20 points) Box I contains Jack, Queen, King card of the same suit and Joker.

(a) *Three* of these 4 cards are selected at random without replacement, and moved to *Box II*. Let B be the event that Box II contains the Joker. Find $P(B)$ and $P(B^c)$.

$P(B) = \frac{\binom{3}{2}}{\binom{4}{3}} = \frac{3}{4}$, and $P(B^c) = 1 - \frac{3}{4} = \frac{1}{4}$.

(b) Now *two cards* are drawn from *Box II* at random without replacement. Let C be the event that two face cards (Jack, Queen, or King) are drawn. Find the conditional probabilities $P(C|B)$ and $P(C|B^c)$.

$P(C|B) = \frac{1}{\binom{3}{2}} = \frac{1}{3}$ and $P(C|B^c) = 1$.

(c) Find $P(C)$.

By applying the law of total probability we obtain

$$P(C) = P(C|B)P(B) + P(C|B^c)P(B^c) = \frac{1}{2},$$

which is equivalently given by the probability that two face cards are drawn directly from Box I, and calculated by

$$\frac{\binom{3}{2}}{\binom{4}{2}} = \frac{1}{2}$$

.

(d) Given that *two face cards* are drawn from *Box II*, find the *conditional probability* that *box II* contains Joker, that is, find $P(B|C)$.

$P(B|C) = \frac{P(C|B)P(B)}{P(C)} = \frac{1}{2}$.

Q.2 (15 points) Let B and D be independent events with $P(B) = \frac{1}{5}$ and $P(D) = \frac{1}{3}$.

(a) Find $P(B \cap D)$.

Since B and D are independent, we have $P(B \cap D) = P(B)P(D) = \frac{1}{15}$

(b) Find $P(B \cup D)$.

$P(B \cup D) = P(B) + P(D) - P(B \cap D) = \frac{1}{5} + \frac{1}{3} - \frac{1}{15} = \frac{7}{15}$

(c) Find the *conditional probability* $P(B|B \cup D)$.

Here you must notice $B \cap (B \cup D) = B$. Then we can find

$$P(B|B \cup D) = \frac{P(B)}{P(B \cup D)} = \frac{1/5}{7/15} = \frac{3}{7}$$

Q.3 (15 points) Let X and Y be two random variables. We know that $E[X] = \frac{1}{2}$, $\mathrm{Var}(X) = \frac{3}{2}$, $E[Y] = 1$, $\mathrm{Var}(Y) = 1$, and $E[XY] = \frac{3}{4}$.

(a) Find $E[2X - 3Y]$.

$E[2X - 3Y] = 2E[X] - 3E[Y] = -2$

(b) Find $\mathrm{Var}(2X+1)$.

$\mathrm{Var}(2X+1) = (2^2)\mathrm{Var}(X) = 6$

(c) Find $\mathrm{Var}(X+Y)$.

$\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y] = \frac{1}{4}$. Thus, $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + 2\mathrm{Cov}(X,Y) + \mathrm{Var}(Y) = 3$.

Q.4 (20 points) Suppose that a continuous random variable X has the following pdf:

$$f(x) = \begin{cases} \alpha(10-x) & \text{if } 0 \le x \le 10; \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the value α.

Since $\displaystyle\int_0^{10} \alpha(10-x)\,dx = 50\alpha = 1$, we obtain $\alpha = 1/50$.

(b) Find the cdf $F(x)$, and sketch the graph of $F(x)$.

$$F(x) = \begin{cases} 0 & \text{if } x < 0; \\ 1 - \left(\frac{x-10}{10}\right)^2 & \text{if } 0 \le x \le 10; \\ 1 & \text{if } 10 < x. \end{cases}$$

(c) Find $P(5 \le X \le 10)$.

$P(5 \le X \le 10) = \displaystyle\int_5^{10} \frac{10-x}{50}\,dx = \frac{1}{4}$.

(d) Find $E[X]$ and $E[30X + 3000]$.

$E[X] = \displaystyle\int_0^{10} x\left(\frac{10-x}{50}\right)\,dx = \frac{10}{3}$.

$E[3000 + 30X] = 3000 + 30E[X] = 3100$.

Q.5 (15 points) A pair (X,Y) of discrete random variables has the joint frequency function

$$p(0,0) = \tfrac{1}{16};\ p(0,1) = \tfrac{1}{16};\ p(0,2) = \tfrac{2}{16};\ p(1,0) = \tfrac{3}{16};\ p(1,1) = \tfrac{1}{16};\ p(1,2) = \tfrac{8}{16}.$$

(a) Find the marginal frequency functions of X and Y, $p_X(0)$, $p_X(1)$, and $p_Y(0)$, $p_Y(1)$, $p_Y(2)$, and confirm that $p(0,0) = p_X(0)p_Y(0)$. Are X and Y independent?

$p_X(0) = \frac{1}{4}$, $p_X(1) = \frac{3}{4}$, and $p_Y(0) = \frac{1}{4}$, $p_Y(1) = \frac{1}{8}$, $p_Y(2) = \frac{5}{8}$. Thus, we find $p(0,0) = p_X(0)p_Y(0) = \frac{1}{16}$, but it is not sufficient for independence; in fact, they are not independent, as justified in (c).

(b) Calculate $E[X]$ and $E[Y]$.

$E[X] = \displaystyle\sum_{x=0}^{1}(x)p_X(x) = \tfrac{3}{4}$; $E[Y] = \sum_{y=0}^{2}(y)p_Y(y) = \tfrac{11}{8}$.

(c) Calculate $E[XY]$. Are X and Y independent? This time justify your answer.

$E[XY] = \displaystyle\sum_{x=0}^{1}\sum_{y=0}^{2}(xy)p(x,y) = \tfrac{17}{16}$. They are not independent since $E[XY] \neq E[X]E[Y]$.

Q.6 (15 points) Let X and Y be continuous random variables. The joint density function of X and Y is given by

$$f(x,y) = \begin{cases} \frac{xy}{4} & \text{if } 0 \le x \le 2 \text{ and } 0 \le y \le 2; \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $P(X \le 1, Y \le 1)$.

$$P(X \le 1, Y \le 1) = \int_0^1 \int_0^1 \frac{xy}{4}\, dx\, dy = \frac{1}{16}.$$

(b) Find the marginal densities of X and Y.

$$f_X(x) = \int_0^2 \frac{xy}{4}\, dy = \frac{x}{2},\ 0 \le x \le 2;$$

$$f_Y(y) = \int_0^2 \frac{xy}{4}\, dx = \frac{y}{2},\ 0 \le y \le 2.$$

(c) Calculate $\mathrm{Cov}(X,Y)$.

Since the joint density function satisfies $f(x,y) = f_X(x)f_Y(y)$, X and Y independent. Thus, we obtain $\mathrm{Cov}(X,Y) = 0$.