

Derivatives and integrals of basic functions.

Functions	Derivatives	Indefinite integrals
Fundamental theorem of calculus	$\frac{d}{dx} F(x) = f(x)$	$\int f(x) dx = F(x) + C$
Power and logarithmic functions	$\frac{d}{dx} x^n = nx^{n-1}$ $\frac{d}{dx} \ln x = \frac{1}{x}$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$ $\int \frac{1}{x} dx = \ln x + C$
Exponential functions	$\frac{d}{dx} e^x = e^x$	$\int e^x dx = e^x + C$
Trigonometric functions	$\frac{d}{dx} \sin x = \cos x$ $\frac{d}{dx} \cos x = -\sin x$ $\frac{d}{dx} \tan x = \frac{1}{\cos^2 x} = \sec^2 x$ $\frac{d}{dx} \cot x = -\frac{1}{\sin^2 x} = -\csc^2 x$ $\frac{d}{dx} \sec x = \sec x \tan x$ $\frac{d}{dx} \csc x = -\csc x \cot x$	$\int \cos x dx = \sin x + C$ $\int \sin x dx = -\cos x + C$ $\int \sec^2 x dx = \int \frac{1}{\cos^2 x} dx = \tan x + C$ $\int \csc^2 x dx = \int \frac{1}{\sin^2 x} dx = -\cot x + C$ $\int \sec x \tan x dx = \sec x + C$ $\int \sec x dx = \ln \sec x + \tan x + C$ $\int \csc x \cot x dx = -\csc x + C$ $\int \csc x dx = -\ln \csc x + \cot x + C$
Hyperbolic functions	$\frac{d}{dx} \sinh x = \cosh x$ $\frac{d}{dx} \cosh x = \sinh x$ $\frac{d}{dx} \tanh x = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$ $\frac{d}{dx} \coth x = -\frac{1}{\sinh^2 x} = -\operatorname{csch}^2 x$	$\int \cosh x dx = \sinh x + C$ $\int \sinh x dx = \cosh x + C$ $\int \frac{1}{\cosh^2 x} dx = \tanh x + C$ $\int \frac{1}{\sinh^2 x} dx = -\coth x + C$
Inverse trigonometric functions	$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$	$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$ $\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$
Inverse hyperbolic functions	$\frac{d}{dx} \sinh^{-1} x = \frac{1}{\sqrt{x^2+1}}$ $\frac{d}{dx} \cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}$ $\frac{d}{dx} \tanh^{-1} x = \frac{1}{1-x^2}$	$\int \frac{1}{\sqrt{x^2+1}} dx = \sinh^{-1} x + C$ $\int \frac{1}{\sqrt{x^2-1}} dx = \cosh^{-1} x + C$ $\int \frac{1}{1-x^2} dx = \tanh^{-1} x + C$

Formulas and identities of basic functions.

Logarithmic and Exponential functions	$\log_a x = \frac{\ln x}{\ln a}$	$a^x = e^{x \ln a}$
Trigonometric functions	$\tan x = \frac{\sin x}{\cos x}$ $\sec x = \frac{1}{\cos x}$ $\csc x = \frac{1}{\sin x}$ $\cot x = \frac{\cos x}{\sin x}$ $\cos^2 x + \sin^2 x = 1$ $1 + \tan^2 x = \sec^2 x$ $1 - \cos^2 x = \sin^2 x$ $\sec^2 - 1 = \tan^2 x$ $\sec^{-1} x = \cos^{-1}(1/x)$ $\csc^{-1} x = \sin^{-1}(1/x)$ $\cot^{-1} x = \tan^{-1}(1/x)^\dagger$ $\sin 2x = 2 \sin x \cos x$ $\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}^\ddagger$ $\cos 2x = \cos^2 x - \sin^2 x$ $\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}^\ddagger$ $\sin^2 x = \frac{1 - \cos 2x}{2}$ $\cos^2 x = \frac{1 + \cos 2x}{2}$ $A \cos x + B \sin x = \sqrt{A^2 + B^2} \cos(x - \theta)$ where $A > 0$ and $\theta = \tan^{-1}(B/A)$.	
Hyperbolic functions	$\sinh x = \frac{e^x - e^{-x}}{2}$ $\cosh x = \frac{e^x + e^{-x}}{2}$ $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ $\operatorname{sech} x = \frac{1}{\cosh x}$ $\operatorname{csch} x = \frac{1}{\sinh x}$ $\coth x = \frac{1}{\tanh x}$ $\cosh^2 x - \sinh^2 x = 1$	$\sinh^{-1} x = \ln \left x + \sqrt{x^2 + 1} \right $ $\cosh^{-1} x = \ln \left x + \sqrt{x^2 - 1} \right $ $\tanh^{-1} x = \frac{1}{2} \ln \left \frac{1+x}{1-x} \right $

† Derivatives of inverse trigonometric functions:

$$(a) \frac{d}{dx} \sec^{-1} x = \frac{d}{dx} \cos^{-1}(1/x) = \frac{1}{x^2 \sqrt{1 - (1/x)^2}} = \frac{1}{|x| \sqrt{x^2 - 1}}$$

$$(b) \frac{d}{dx} \cot^{-1} x = \frac{d}{dx} \tan^{-1}(1/x) = -\frac{1}{x^2(1 + (1/x)^2)} = -\frac{1}{x^2 + 1}$$

‡ Half-angle tangent substitution:

Substitution	Identities
$t = \tan\left(\frac{x}{2}\right), \quad \frac{x}{2} = \tan^{-1} t;$	$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad dx = \frac{2}{1+t^2} dt.$

Example. $\int \frac{1 + \sin x}{1 + \cos x} dx = \int \frac{(1+t)^2}{1+t^2} dt = \int \left(1 + \frac{2t}{1+t^2}\right) dt = t + \int \frac{1}{s} ds \quad \boxed{(A.1) \ s = 1 + t^2, \ \frac{ds}{dt} = 2t}$

$$= t + \ln |s| + C = \tan \frac{x}{2} + \ln \left| 1 + \tan^2 \frac{x}{2} \right| + C$$

Strategies for integration and sample problems.

No silver bullet. Here is the review of basic integration techniques, together with some of the challenging problems presented at *University of North Texas Integration Bee*. You will see that even computer algebra systems such as Maple and Mathematica are not tough enough to complete them. Nevertheless, you could outperform smart software with the basic techniques we have learned.

Simplifying the integrand first. Identities can be used to fit the integrand to the integral of basic functions.

$$(a) \int \frac{1}{1 + \cos 2x} dx = \int \frac{1}{2 \cos^2 x} dx = \frac{1}{2} \tan x + C$$

$$(b) \int \frac{1}{\sin x + \cos x} dx = \frac{1}{\sqrt{2}} \int \frac{1}{\cos(x - \frac{\pi}{4})} dx = \frac{1}{\sqrt{2}} \int \sec\left(x - \frac{\pi}{4}\right) dx$$

$$= \frac{1}{\sqrt{2}} \ln \left| \sec\left(x - \frac{\pi}{4}\right) + \tan\left(x - \frac{\pi}{4}\right) \right| + C \quad (\text{Mathematica cannot evaluate; Maple can.})$$

A. Substitution rules.

$$(A.1) \int y \frac{dt}{dx} dx = \int y dt$$

$$(a) \int \frac{\sin \sqrt{x}}{\sqrt{x}} dx = 2 \int \sin t dt \quad \boxed{(A.1) \ t = \sqrt{x}, \ \frac{dt}{dx} = \frac{1}{2\sqrt{x}}}$$

$$= -2 \cos t + C = -2 \cos \sqrt{x} + C$$

$$(b) \int \frac{\cos(\ln x)}{x} dx = \int \cos t dt \quad \boxed{(A.1) \ t = \ln x, \ \frac{dt}{dx} = \frac{1}{x}}$$

$$= \sin t + C = \sin(\ln x) + C$$

$$(c) \int \frac{\sin x}{1 + \sin x} dx = \int \frac{\sin x(1 - \sin x)}{1 - \sin^2 x} dx$$

$$= \int \frac{\sin x}{\cos^2 x} dx - \int \frac{\sin^2 x}{\cos^2 x} dx = - \int \frac{1}{t^2} dt - \int \left[\frac{1}{\cos^2 x} - 1 \right] dx \quad \boxed{(A.1) \ t = \cos x, \ \frac{dt}{dx} = -\sin x}$$

$$= \frac{1}{t} - [\tan x - x] + C = \frac{1}{\cos x} - \tan x + x + C$$

$$(d) \int \frac{1}{\sqrt{1 - x^2 + \sin^{-1} x - x^2 \sin^{-1} x}} dx \quad (\text{Maple cannot evaluate; Mathematica can.})$$

$$= \int \frac{1}{\sqrt{(1 - x^2)(1 + \sin^{-1} x)}} dx = \int \frac{1}{\sqrt{t}} dt \quad \boxed{(A.1) \ t = 1 + \sin^{-1} x, \ \frac{dt}{dx} = \frac{1}{\sqrt{1 - x^2}}}$$

$$= 2t^{\frac{1}{2}} + C = 2\sqrt{1 + \sin^{-1} x} + C$$

$$(A.2) \int y dx = \int y \frac{dx}{dt} dt$$

$$(a) \int x^2 \sqrt{x+4} dx = \int (t^2 - 4)^2 t 2t dt \quad \boxed{(A.2) \ t^2 = x + 4, \ \frac{dx}{dt} = 2t}$$

$$= 2 \int (t^6 - 8t^4 + 16t^2) dt = 2 \left(\frac{1}{7} t^7 - \frac{8}{5} t^5 + \frac{16}{3} t^3 \right) + C$$

$$= \frac{2}{7} (x+4)^{\frac{7}{2}} - \frac{16}{5} (x+4)^{\frac{5}{2}} + \frac{32}{3} (x+4)^{\frac{3}{2}} + C$$

$$\begin{aligned}
 \text{(b)} \quad \int \sqrt{1-e^x} dx &= \int t \frac{2t}{t^2-1} dt && \boxed{\text{(A.2)} \quad t^2 = 1 - e^x, \quad \frac{dx}{dt} = \frac{2t}{t^2-1}} \\
 &= 2 \int \left[1 - \frac{1}{1-t^2} \right] dt = 2 [t - \tanh^{-1} t] + C = 2\sqrt{1-e^x} - 2 \tanh^{-1} \sqrt{1-e^x} + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad \int \frac{1}{x^4-16} dx &= \frac{1}{8} \int \frac{1}{t^4-1} dt && \boxed{\text{(A.2)} \quad x = 2t, \quad \frac{dx}{dt} = 2} \\
 &= \frac{1}{16} \int \left[\frac{1}{t^2-1} - \frac{1}{t^2+1} \right] dx = \frac{1}{16} [-\tanh^{-1} t - \tan^{-1} t] + C \\
 &= -\frac{1}{16} \tanh^{-1} \left(\frac{x}{2} \right) - \frac{1}{16} \tan^{-1} \left(\frac{x}{2} \right) + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad \int \frac{x^2}{\sqrt{x-1}} dx &= 2 \int (t^2+1)^2 dt && \boxed{\text{(A.2)} \quad t^2 = x-1, \quad \frac{dx}{dt} = 2t} \\
 &= 2 \int [t^4 + 2t^2 + 1] dt = \frac{2}{5} t^5 + \frac{4}{3} t^3 + 2t + C = \frac{2}{5} (x-1)^{\frac{5}{2}} + \frac{4}{3} (x-1)^{\frac{3}{2}} + 2(x-1)^{\frac{1}{2}} + C
 \end{aligned}$$

$$\text{(A.3)} \quad \int \frac{1}{y} \frac{dy}{dx} dx = \ln |y| + C$$

$$\begin{aligned}
 \text{(a)} \quad \int \sec x dx &= \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} dx && \boxed{\text{(A.3)} \quad y = \sec x + \tan x, \quad \frac{dy}{dx} = \sec x \tan x + \sec^2 x} \\
 &= \ln |y| + C = \ln |\sec x + \tan x| + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \int \csc x dx &= \int \csc x \frac{\csc x + \cot x}{\csc x + \cot x} dx && \boxed{\text{(A.3)} \quad y = \csc x + \cot x, \quad \frac{dy}{dx} = -\csc x \cot x - \csc^2 x} \\
 &= -\ln |y| + C = -\ln |\csc x + \cot x| + C
 \end{aligned}$$

(A.4) Trigonometric substitution.

Substitution	Identities
$x = a \sin t, \quad t = \sin^{-1} \left(\frac{x}{a} \right);$	$a^2 - x^2 = a^2 \cos^2 t, \quad dx = a \cos t dt.$
$x = a \tan t \quad t = \tan^{-1} \left(\frac{x}{a} \right);$	$a^2 + x^2 = a^2 \sec^2 t, \quad dx = a \sec^2 t dt.$
$x = a \sec t \quad t = \cos^{-1} \left(\frac{a}{x} \right);$	$x^2 - a^2 = a^2 \tan^2 t, \quad dx = a \sec t \tan t dt.$

$$\text{(a)} \quad \int \frac{1}{\sqrt{1-x^2}} dx = \int dt = t + C = \sin^{-1} x + C \quad \boxed{\text{(A.4)} \quad x = \sin t}$$

$$\text{(b)} \quad \int \frac{1}{1+x^2} dx = \int dt = t + C = \tan^{-1} x + C \quad \boxed{\text{(A.4)} \quad x = \tan t}$$

$$\begin{aligned}
 \text{(c)} \quad \int \frac{1}{1-x^2} dx &= \int \sec t dt && \boxed{\text{(A.4)} \quad x = \sin t} \\
 &= \ln |\sec t + \tan t| + C = \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right| = \tanh^{-1} x + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad \int \frac{1}{\sqrt{1+x^2}} dx &= \int \sec t \, dt && \boxed{\text{(A.4)} \ x = \tan t} \\
 &= \ln |\sec t + \tan t| + C = \ln \left| \sqrt{1+x^2} + x \right| = \sinh^{-1} x + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(e)} \quad \int \frac{1}{\sqrt{x^2-1}} dx &= \int \sec t \, dt && \boxed{\text{(A.4)} \ x = \sec t} \\
 &= \ln |\sec t + \tan t| + C = \ln \left| x + \sqrt{x^2-1} \right| = \cosh^{-1} x + C
 \end{aligned}$$

B. Integration by parts: $\int u \, dv = uv - \int v \, du$

$$\begin{aligned}
 \text{(a)} \quad \int \sqrt{x} \ln x \, dx &= \frac{2}{3} x^{\frac{3}{2}} \ln x - \frac{2}{3} \int x^{\frac{1}{2}} dx && \boxed{\text{(B)} \ u = \ln x, \ v = \frac{2}{3} x^{\frac{3}{2}}} \\
 &= \frac{2}{3} x^{\frac{3}{2}} \ln x - \frac{4}{9} x^{\frac{3}{2}} + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \int e^x \sin x \, dx &= e^x \sin x - \int e^x \cos x \, dx && \boxed{\text{(B)} \ u = \sin x, \ v = e^x} && \text{(Maple cannot simplify.)} \\
 &= e^x \sin x - \left[e^x \cos x + \int e^x \sin x \, dx \right] && \boxed{\text{(B)} \ u = \cos x, \ v = e^x} \\
 &= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx, \text{ which implies } \int e^x \sin x \, dx = \frac{e^2}{2} (\sin x - \cos x).
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad \int x^3 e^{-x^2} dx &= \frac{1}{2} \int t e^{-t} dt && \boxed{\text{(A.1)} \ t = x^2, \ \frac{dt}{dx} = 2x} \\
 &= \frac{1}{2} \left[-t e^{-t} + \int e^{-t} dt \right] && \boxed{\text{(B)} \ u = t, \ v = -e^{-t}} \\
 &= \frac{1}{2} \left[-t e^{-t} + \int e^{-t} dt \right] = \frac{1}{2} [-t e^{-t} - e^{-t}] + C = -\frac{1}{2} x^2 e^{-x^2} - \frac{1}{2} e^{-x^2} + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad \int \frac{x}{2+e^x+e^{-x}} dx &= 4 \int \frac{t}{(e^t+e^{-t})^2} dt && \boxed{\text{(A.2)} \ x = 2t, \ \frac{dx}{dt} = 2} \\
 &= \int \frac{t}{\cosh^2 t} dt = t \tanh t - \int \tanh t \, dt && \boxed{\text{(B)} \ u = t, \ v = \tanh t} \\
 &= t \tanh t - \int \frac{\sinh t}{\cosh t} dt = t \tanh t - \int \frac{1}{s} ds && \boxed{\text{(A.1)} \ s = \cosh t, \ \frac{ds}{dt} = \sinh t} \\
 &= t \tanh t - \ln |s| + C = (x/2) \tanh(x/2) - \ln |\cosh(x/2)| + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(e)} \quad \int x^3 \sin x^2 dx &= \frac{1}{2} \int t \sin t \, dt && \boxed{\text{(A.1)} \ t = x^2, \ \frac{dt}{dx} = 2x} \\
 &= \frac{1}{2} \left[-t \cos t + \int \cos t \, dt \right] && \boxed{\text{(B)} \ u = t, \ v = -\cos t} \\
 &= -\frac{t}{2} \cos t + \frac{1}{2} \sin t + C = -\frac{x^2}{2} \cos x^2 + \frac{1}{2} \sin x^2 + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(f)} \quad \int \sin x \ln |\cos x| dx &= \int -\ln |t| dt && \boxed{\text{(A.1)} \quad t = \cos x, \quad \frac{dt}{dx} = -\sin x} \\
 &= -t \ln |t| + \int dt && \boxed{\text{(B)} \quad u = \ln |t|, \quad v = -t} \\
 &= -t \ln |t| + t + C = -\cos x \ln |\cos x| + \cos x + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(g)} \quad \int \frac{x^2}{(x^2 + 8)^{\frac{3}{2}}} dx &= -x(x^2 + 8)^{\frac{1}{2}} + \int \frac{1}{(x^2 + 8)^{\frac{1}{2}}} dx && \boxed{\text{(B)} \quad u = x, \quad v = -(x^2 + 8)^{\frac{1}{2}}} \\
 &= -\frac{x}{\sqrt{x^2 + 8}} + \int \frac{1}{\sqrt{t^2 + 1}} dt && \boxed{\text{(A.2)} \quad x = \sqrt{8}t, \quad \frac{dx}{dt} = \sqrt{8}} \\
 &= -\frac{x}{\sqrt{x^2 + 8}} + \sinh^{-1} t + C = -\frac{x}{\sqrt{x^2 + 8}} + \sinh^{-1} \left(\frac{x}{\sqrt{8}} \right) + C
 \end{aligned}$$

C. Integration of rational functions.

(C.1) Partial fractions

$$\begin{aligned}
 \frac{1}{x^3 + 1} &= \frac{1}{3} \left[\frac{1}{x + 1} - \frac{x - 2}{x^2 - x + 1} \right]; & \frac{1}{x^3 - 1} &= \frac{1}{3} \left[\frac{1}{x - 1} - \frac{x + 2}{x^2 + x + 1} \right]. \\
 \frac{1}{x^4 - 1} &= \frac{1}{2} \left[\frac{1}{x^2 - 1} - \frac{1}{x^2 + 1} \right]; & \frac{1}{x^4 + 1} &= \frac{\sqrt{2}}{4} \left[\frac{x + \sqrt{2}}{x^2 + \sqrt{2}x + 1} - \frac{x - \sqrt{2}}{x^2 - \sqrt{2}x + 1} \right]^* \\
 & \left[* \text{ Use } x^4 + 1 = (x^2 + 1)^2 - (\sqrt{2}x)^2 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1) \right]
 \end{aligned}$$

$$\begin{aligned}
 \text{(a)} \quad \int \frac{1}{x^3 + 8} dx &= \frac{1}{4} \int \frac{1}{t^3 + 1} dx && \boxed{\text{(A.2)} \quad x = 2t, \quad \frac{dx}{dt} = 2} \\
 &= \frac{1}{12} \int \left(\frac{1}{t + 1} - \frac{t - 2}{t^2 - t + 1} \right) dt && \boxed{\text{(C.1)}} \\
 &= \frac{1}{12} \ln |t + 1| - \frac{1}{12} \left[\frac{1}{2} \ln |t^2 - t + 1| - \frac{3}{2} \int \frac{1}{t^2 - t + 1} dt \right] && \boxed{\text{(C.2)}} \\
 &= \frac{1}{12} \ln |t + 1| - \frac{1}{24} \ln |t^2 - t + 1| + \frac{\sqrt{3}}{12} \tan^{-1} \left(\frac{2t - 1}{\sqrt{3}} \right) + C && \boxed{\text{(C.3)}} \\
 &= \frac{1}{12} \ln \left| \frac{x}{2} + 1 \right| - \frac{1}{24} \ln \left| \frac{x^2}{4} - \frac{x}{2} + 1 \right| + \frac{\sqrt{3}}{12} \tan^{-1} \left(\frac{x - 1}{\sqrt{3}} \right) + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(C.2)} \quad \int \frac{x + a}{x^2 + bx + c} dx &= \left(\frac{1}{2} \right) \ln |x^2 + bx + c| + \left(a - \frac{b}{2} \right) \int \frac{1}{x^2 + bx + c} dx \\
 & \left[* \text{ Use } \frac{x+a}{x^2+bx+c} = \left(\frac{1}{2} \right) \frac{2x+b}{x^2+bx+c} + \left(a - \frac{b}{2} \right) \frac{1}{x^2+bx+c} \text{ and (A.1) with } t = x^2 + bx + c, \quad \frac{dt}{dx} = 2x + b \right]
 \end{aligned}$$

$$\text{(C.3)} \quad \int \frac{1}{x^2 + bx + c} dx = \begin{cases} \frac{2}{|b^2 - 4c|^{\frac{1}{2}}} \tan^{-1} \left(\frac{2x+b}{|b^2 - 4c|^{\frac{1}{2}}} \right)^{**} & \text{if } b^2 - 4c < 0; \\ -\frac{2}{|b^2 - 4c|^{\frac{1}{2}}} \tanh^{-1} \left(\frac{2x+b}{|b^2 - 4c|^{\frac{1}{2}}} \right)^{**} & \text{if } b^2 - 4c > 0. \end{cases}$$

$$\left[** \text{ Use } \frac{1}{x^2+bx+c} = \frac{4}{(2x+b)^2 - (b^2 - 4c)} \text{ and (A.2) with } |b^2 - 4c|^{\frac{1}{2}} t = 2x + b, \quad \frac{dx}{dt} = \frac{|b^2 - 4c|^{\frac{1}{2}}}{2} \right]$$

$$(a) \int \frac{1}{x^2 + x + 1} dx = \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2x+1}{\sqrt{3}} \right) + C \quad \boxed{\text{(C.3)}}$$

$$\begin{aligned} (b) \int \frac{x}{x^4 + x^2 + 1} dx &= \int \frac{x}{(x^2 + \frac{1}{2})^2 + \frac{3}{4}} dx = \frac{1}{2} \int \frac{1}{t^2 + \frac{3}{4}} dt & \boxed{\text{(A.1) } t = x^2 + \frac{1}{2}, \frac{dt}{dx} = 2x} \\ &= \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{2t}{\sqrt{3}} \right) + C & \boxed{\text{(C.3)}} \\ &= \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{2x^2 + 1}{\sqrt{3}} \right) + C \end{aligned}$$

$$\begin{aligned} \text{(C.4)} \int \frac{1}{(1+x^2)^{n+1}} dx &= \frac{x}{2n(1+x^2)^n} + \frac{2n-1}{2n} \int \frac{1}{(1+x^2)^n} dx^{***} \\ \int \frac{1}{(1-x^2)^{n+1}} dx &= \frac{x}{2n(1-x^2)^n} + \frac{2n-1}{2n} \int \frac{1}{(1-x^2)^n} dx \\ \left[\text{*** Use } \frac{d}{dx} \left(\frac{x}{(1+x^2)^n} \right) &= -\frac{2n-1}{(1+x^2)^n} + \frac{2n}{(1+x^2)^{n+1}} \right] \end{aligned}$$

$$\begin{aligned} (a) \int \sec^5 x dx &= \int \frac{1}{\cos^5 x} dx = \int \frac{\cos x}{(1-\sin^2 x)^3} dx = \int \frac{1}{(1-t^2)^3} dt & \boxed{\text{(A.1) } t = \sin x, \frac{dt}{dx} = \cos x} \\ &= \frac{t}{4(1-t^2)^2} + \frac{3}{4} \int \frac{1}{(1-t^2)^2} dt = \frac{t}{4(1-t^2)^2} + \frac{3}{4} \left[\frac{t}{2(1-t^2)} + \frac{1}{2} \int \frac{1}{(1-t^2)} dt \right] & \boxed{\text{(C.4) twice}} \\ &= \frac{t}{4(1-t^2)^2} + \frac{3t}{8(1-t^2)} + \frac{3}{8} \tanh^{-1} t + C = \frac{\sin x}{4\cos^4 x} + \frac{3\sin x}{8\cos^2 x} + \frac{3}{8} \tanh^{-1} \sin x + C \end{aligned}$$

$$\begin{aligned} (b) \int \frac{1}{(x^2+4)^3} dx &= \frac{1}{32} \int \frac{1}{(t^2+1)^3} dt & \boxed{\text{(A.2) } x = 2t, \frac{dx}{dt} = 2} \\ &= \frac{1}{32} \left[\frac{t}{4(t^2+1)^2} + \frac{3}{4} \int \frac{1}{(t^2+1)^2} dt \right] = \frac{1}{32} \left[\frac{t}{4(t^2+1)^2} + \frac{3}{4} \left[\frac{t}{2(t^2+1)} + \frac{1}{2} \int \frac{1}{t^2+1} dt \right] \right] & \boxed{\text{(C.4) twice}} \\ &= \frac{1}{32} \left[\frac{t}{4(t^2+1)^2} + \frac{3t}{8(t^2+1)} + \frac{3}{8} \tan^{-1} t \right] + C = \frac{x}{16(x^2+4)^2} + \frac{3x}{128(x^2+4)} + \frac{3}{256} \tan^{-1} \frac{x}{2} + C \end{aligned}$$