Q.1 (20 points) An urn contains three red and two white balls. A ball is drawn, and then it and another ball of the same color are placed back in the urn. Finally, a second ball is drawn. Let A be the event that the first ball drawn is white, and let B be the event that the second ball drawn is white.

(a) Find $P(A \cap B)$.

$P(A \cap B) = P(B|A)\, P(A) = \left(\frac{3}{6}\right) \left(\frac{2}{5}\right) = \frac{1}{5}$

(b) Find $P(B)$.

$P(B) = P(B|A)\, P(A) + P(B|A^c)\, P(A^c) = \left(\frac{3}{6}\right) \left(\frac{2}{5}\right) + \left(\frac{2}{6}\right) \left(\frac{3}{5}\right) = \frac{2}{5}$

(c) Find $P(A|B)$.

$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{1}{2}$

Q.2 (20 points) Let X and Y be two random variables. Suppose that (i) X and Y are independent, (ii) $E[X] = 3$ and $E[Y] = -2$, and (iii) $E[X^2] = 13$ and $E[Y^2] = 5$. Then answer the following questions.

(a) Find $E[3X + 2Y]$.

$E[3X + 2Y] = (3)(3) + (2)(-2) = 5$

(b) Find $\mathrm{Var}(X)$.

$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = 13 - (3)^2 = 4$

(c) Find $\mathrm{Var}(3X)$.

$\mathrm{Var}(3X) = (3^2)\mathrm{Var}(X) = (9)(4) = 36.$

(d) Find $\mathrm{Var}(3X + 2Y)$.

Since $\mathrm{Var}(Y) = E[Y^2] - (E[Y])^2 = 5 - (-2)^2 = 1$, we obtain

$$\mathrm{Var}(3X + 2Y) = (3^2)\mathrm{Var}(X) + (2^2)\mathrm{Var}(Y) = (9)(4) + (4)(1) = 40$$

Q.3 (20 points) Suppose that a rare disease has an incidence of 1 in 1000. Assume that members of a population of $10,000$ are affected independently. Answer the following questions.

(a) What distribution do we apply to approximate the number of disease cases? Identify parameter(s) if any.

Poisson distribution with $\lambda = (0.001)(10,000) = 10$.

(b) The probability of one case in the population is approximated by ce^{-c}. Identify the value c.

We obtain $p(1) = \lambda e^{-\lambda} = 10e^{-10}$. Thus, $c = 10$.

(c) The probability of two cases in the population is approximated by be^{-c}. Identify the value b.

We obtain $p(2) = e^{-\lambda}\dfrac{\lambda^2}{2!} = 50e^{-10}$. Thus, $b = 50$.

Q.4 (20 points) An actual voltage of 3-volt battery has the probability density function

$$f(x) = \frac{1}{6}, \quad 0 \le x \le 6.$$

Answer the following questions.

(a) Let X be a random variable having the pdf $f(x)$. Find the mean and standard deviation of X.

$E[X] = 3$ and $\sqrt{\mathrm{Var}(X)} = \sqrt{3}$.

(b) Let $\bar{X}$ be the average voltage from 48 batteries. What distribution do you apply to approximate $\bar{X}$? Identify parameter(s) if any.

Normal distribution with $\mu = 3$ and $\sigma^2 = 1/16$ (that is, $\sigma = 1/4$)

(c) What is the probability that $|\bar{X} - 3| \leq 0.5$?

$$P(2.5 \leq \bar{X} \leq 3.5) = \Phi(2.0) - \Phi(-2.0) = 0.9772 - 0.0228 = 0.9544.$$

(d) Estimate the probability that the sum of the voltages from 48 batteries lies between 138 and 156 volts.

The distribution for the sum Y of 48 voltages is approximated by a normal distribution with $\mu = 144$ and $\sigma^2 = 144$ (that is, $\sigma = 12$). Thus, we obtain

$$P(138 \leq Y \leq 156) = \Phi(1.0) - \Phi(-0.5) = 0.8413 - 0.3085 = 0.5328.$$

Q.5 (20 points) A study shows that the probability of a binge drink is 0.2 among college students. Let X be the number of students who binge drink out of sample size 25. Answer the following questions.

(a) Describe the exact distribution for X. Identify parameter(s) if any.

Binomial distribution with $n = 25$ and $p = 0.2$.

(b) Find the exact probability $P(X = 0)$, and express it in a form of a^n.

$P(X = 0) = (0.8)^{25}$

(c) Find the mean and standard deviation of X.

$E[X] = np = 5$ and $\sqrt{\mathrm{Var}(X)} = \sqrt{np(1-p)} = 2$.

(d) By using the normal approximation with continuity correction, find $P(4 \leq X \leq 8)$.

$$P(4 \leq X \leq 8) = \Phi\left(\frac{8.5 - 5}{2}\right) - \Phi\left(\frac{3.5 - 5}{2}\right) = \Phi(1.75) - \Phi(-0.75) = 0.9599 - 0.2266 = 0.7333.$$