Conditional Probability and Independent Events

Conditional probability. Let A and B be two events where $P(B) \neq 0$. Then the **conditional probability** of A given B can be defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

The idea of "conditioning" is that "if we have known that B has occurred, the sample space should have become B rather than Ω."

Example 1. Mr. and Mrs. Jones have two children. What is the conditional probability that their children are both boys, given that they have at least one son?

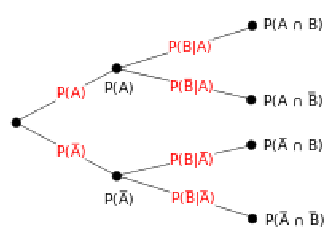
Solution. Let $\Omega = \{(B, B), (B, G), (G, B), (G, G)\}$ be the sample space. Then we can introduce the event $A = \{(B, B)\}$ of both boys, and $B = \{(B, B), (B, G), (G, B)\}$ representing that at least one of them is a boy. Thus, we can find the conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{3/4} = \frac{1}{3}$$

Multiplication rule. If $P(A)$ and $P(B|A)$ are known then we can use them to find

$$P(A \cap B) = P(B|A)P(A).$$

The various forms of multiplication rule are illustrated in the **tree diagram**. The probability that the series of events leading to a particular node will occur is equal to the product of their probabilities.



Example 2. Celine estimates that her chance to receive an A grade would be $\frac{2}{3}$ in a chemistry course, and $\frac{1}{2}$ in a French course. She decides whether to take a French course or a chemistry course this semester based on the flip of a coin.

(a) What is the probability that she gets an A in chemistry?

(b) What is the probability that she gets an A in French?

Solution. Let A be the event of choosing the chemistry course, and $\bar{A}$ be the event of choosing the French course. Let B be the event of receiving an A. Then we know that $P(A) = P(\bar{A}) = \frac{1}{2}$ and $P(B|A) = \frac{2}{3}$ and $P(B|\bar{A}) = \frac{1}{2}$

(a) $P(A \cap B) = P(B|A)P(A) = \left(\frac{2}{3}\right)\left(\frac{1}{2}\right) = \frac{1}{3}$

(b) $P(\bar{A} \cap B) = P(B|\bar{A})P(\bar{A}) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$

Law of total probability. Let A and B be two events. Then we can write the probability $P(B)$ as

$$P(B) = P(B \cap A) + P(B \cap A^c) = P(B|A)P(A) + P(B|A^c)P(A^c).$$

In general, suppose that we have a sequence $A_1, A_2, \ldots, A_n$ of mutually disjoint events satisfying $\bigcup_{i=1}^n A_i = \Omega$, where "mutual disjointness" means that $A_i \cap A_j = \emptyset$ for all $i \neq j$. (The events $A_1, A_2, \ldots, A_n$ are called "a partition of Ω.") Then for any event B we have

$$P(B) = \sum_{i=1}^{n} P(B \cap A_i) = \sum_{i=1}^{n} P(B|A_i)P(A_i).$$

Example 3. An study shows that an accident-prone person will have an accident in one year period with probability 0.4, whereas this probability is only 0.2 for a non-accident-prone person. Suppose that we assume that 30 percent of all the new policyholders of an insurance company is accident prone. Then what is the probability that a new policyholder will have an accident within a year of purchasing a policy?

Solution. Let A be the event that a policy holder is accident-prone, and let B be the event that the policy holder has an accident within a year. Then we know that $P(A) = 0.3$, $P(A^c) = 0.7$, $P(B|A) = 0.4$, and $P(B|A^c) = 0.2$. Hence, we obtain

$$P(B) = P(B|A)P(A) + P(B|A^c)P(A^c) = (0.4)(0.3) + (0.2)(0.7) = 0.26$$

Independent events. Intuitively we would like to say that A and B are independent if knowing about one event give us no information about another. That is, $P(A|B) = P(A)$ and $P(B|A) = P(B)$. We say A and B are **independent** if

$$P(A \cap B) = P(A)P(B).$$

This definition is symmetric in A and B, and allows $P(A)$ and/or $P(B)$ to be 0. Furthermore, a collection of events $A_1, A_2, \ldots, A_n$ is said to be **mutually independent** if it satisfies

$$P(A_{i_1} \cap \cdots \cap A_{i_m}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_m})$$

for any subcollection $A_{i_1}, A_{i_2}, \ldots, A_{i_m}$.

Example 4. A coin is tossed three times. Then let A be the event that the first and second tosses match, and let B be the event the second and third tosses match.

(a) Define the sample space.

(b) Express A, B and $A \cap B$ as subsets of Ω.

(c) Are A and B independent?

Solution.

(a) $\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$,

(b) $A = \{HHH, HHT, TTT, TTH\}$, $B = \{HHH, THH, TTT, HTT\}$, and $A \cap B = \{HHH, TTT\}$.

(c) $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{2}$, and $P(A \cap B) = \frac{1}{4}$. Thus, A and B satisfy $P(A \cap B) = P(A)P(B)$, and therefore, they are independent.

Assignment No.2

Supplementary Readings.

SS: Murray R. Spiegel, John Schiller, and R. Alu Srinivasan, *Probability and Statistics 4th ed.* McGraw-Hill.

Chapter 1: Conditional Probability, Theorems on Conditional Probability, Independent Events, Bayes Theorem.

TH: Elliot A. Tanis and Robert V. Hogg, *A Brief Course in Mathematical Statistics.* Prentice Hall.

Section 1.3–1.4

WM: Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, and Keying Ye, *Probability & Statistics for Engineers & Scientists*, 9th ed. Prentice Hall.

Section 2.6.

Problem 1. A couple has two children.

(a) What is the probability that both are girls given that the oldest is a girl?

(b) What is the probability that both are girls given that at least one of them is a girl?

Problem 2. Urn 1 has three red balls and two white balls, and urn 2 has two red balls and five white balls. A fair coin is tossed; if it lands heads up, a ball is drawn from urn 1, and otherwise, a ball is drawn from urn 2.

(a) What is the probability that a red ball is drawn?

(b) If a red ball is drawn, what is the probability that the coin landed heads up?

Problem 3. An urn contains three red and two white balls. A ball is drawn, and then it and another ball of the same color are placed back in the urn. Finally, a second ball is drawn.

(a) What is the probability that the second ball drawn is white?

(b) If the second ball drawn is white, what is the probability that the first ball drawn was red?

Problem 4. Let A and B be independent events with $P(A) = 0.7$ and $P(B) = 0.2$. Compute the following probabilities:

(a) $P(A \cap B)$

(b) $P(A \cup B)$

(c) $P(A^c \cup B^c)$

Problem 5. A die is rolled six times. If the face numbered j is the outcome on the j-th roll, we say "a match has occurred." You win if at least one match occurs during the six trials.

(a) Let A_i be the event that i is observed on the i-th roll. Find $P(A_i)$ and $P(A_i^c)$ for each $i = 1, 2, 3, 4, 5, 6$.

(b) Find $P(A_1^c \cap A_2^c \cap A_3^c \cap A_4^c \cap A_5^c \cap A_6^c)$

(c) Let B be the event that at least one match occurs. Express B^c in terms of A_1, A_2, A_3, A_4, A_5, and A_6.

(d) Find the probability that you win.

Computer project. Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a prize; behind the others, goats. You pick a door, say No.1, and the show's host, Monty Hall, who knows what's behind the doors, opens another door, say No.3, which has a goat. He then says to you, "Do you want to pick door No.2?" Now you will respond to him according to one of the following strategies:

I. Stick with the one you have chosen. Thus, say "no" to Monty.

II. Switch it. Thus, say "yes" to Monty.

III. Flip a coin to decide it. Thus, say "yes" with probability 1/2 (and "no" with probability 1/2).

Computer project, continued. Simulate the show's outcome at least 1000 times for each strategy, I, II, and III, in order to determine which strategy you should adopt for the best result. Calculate the actual probability of winning for each strategy by completing the following questions.

(a) Let A be the event that you pick the door with prize at the first time. Find $P(A)$ and $P(A^c)$

(b) Let B be the event that you choose the door with prize at the second time after Monty shows another door with a goat. Determine $P(B|A)$ and $P(B|A^c)$ for each strategy, I, II, and III. Hint: $P(B|A) = 1$ and $P(B|A^c) = 0$ for strategy I.

(c) Calculate $P(B)$ for each strategy, and show that the simulation result supports your calculation.

Answers

Problem 1. Let $\Omega = \{(B,B),(B,G),(G,B),(G,G)\}$ be the sample space.

(a) Here we can introduce the event $A = \{(G,G)\}$ of both girls, and $B = \{(G,G),(B,G)\}$ representing that the older is a girl. Thus, we obtain

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{1/2} = \frac{1}{2}$$

(b) Here we can introduce the event $A = \{(G,G)\}$ of both girls, and $C = \{(G,G),(B,G),(G,B)\}$ representing that at least one of them is a girl. Then we obtain

$$P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{1/4}{3/4} = \frac{1}{3}$$

Problem 2. Let A be the event that a coin lands head up, and let B be the event that a red ball is drawn. Then we know that $P(A) = P(A^c) = \frac{1}{2}$, $P(B|A) = \frac{3}{5}$, and $P(B|A^c) = \frac{2}{7}$.

(a) We can apply the law of total probability, and calculate $P(B)$ by

$$P(B|A)P(A) + P(B|A^c)P(A^c) = \left(\frac{3}{5}\right)\left(\frac{1}{2}\right) + \left(\frac{2}{7}\right)\left(\frac{1}{2}\right) = \frac{31}{70}$$

(b) We now use the definition of conditional probability, and obtain

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)} = \frac{(3/5)(1/2)}{(31/70)} = \frac{21}{31}$$

Problem 3. Let A be the event that the first ball is red, and let B be the event that the second ball is white. Then we know that $P(A) = \frac{3}{5}$, $P(A^c) = \frac{2}{5}$, $P(B|A) = \frac{2}{6} = \frac{1}{3}$, and $P(B|A^c) = \frac{3}{6} = \frac{1}{2}$.

(a) We can apply the law of total probability, and calculate $P(B)$ by

$$P(B|A)P(A) + P(B|A^c)P(A^c) = \left(\frac{1}{3}\right)\left(\frac{3}{5}\right) + \left(\frac{1}{2}\right)\left(\frac{2}{5}\right) = \frac{2}{5}$$

(b) We now use the definition of conditional probability, and obtain

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)} = \frac{(1/3)(3/5)}{(2/5)} = \frac{1}{2}$$

Problem 4.

(a) $P(A \cap B) = P(A)P(B) = (0.7)(0.2) = 0.14$

(b) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = (0.7) + (0.2) - (0.14) = 0.76$

(c) We must use the fact that A^c and B^c are independent, and obtain

$$\begin{aligned} P(A^c \cup B^c) &= P(A^c) + P(B^c) - P(A^c \cap B^c) \\ &= P(A^c) + P(B^c) - P(A^c)P(B^c) \\ &= (0.3) + (0.8) - (0.3)(0.8) = 0.86 \end{aligned}$$

Problem 5.

(a) $P(A_i) = \frac{1}{6}$ and $P(A_i^c) = 1 - P(A_i) = \frac{5}{6}$ for each $i = 1, 2, 3, 4, 5, 6$.

(b) Since they are independent, we have

$$\begin{aligned} & P(A_1^c \cap A_2^c \cap A_3^c \cap A_4^c \cap A_5^c \cap A_6^c) \\ & = P(A_1^c)P(A_2^c)P(A_3^c)P(A_4^c)P(A_5^c)P(A_6^c) = \left(\frac{5}{6}\right)^6 \end{aligned}$$

(c) The complement B^c represents the event that no matches occur. Thus, we have

$$B^c = A_1^c \cap A_2^c \cap A_3^c \cap A_4^c \cap A_5^c \cap A_6^c$$

(d) $P(B) = 1 - P(B^c) = 1 - \left(\frac{5}{6}\right)^6$