Continuous Probability Distributions

Continuous random variables. A **continuous random variable** is a random variable whose possible values are real values such as 78.6, 5.7, 10.24, and so on. Examples of continuous random variables include temperature, height, diameter of metal cylinder, etc. In what follows, a random variable means a "continuous" random variable, unless it is specifically said to be discrete. The **probability distribution** of a random variable X specifies how its values are distributed over the real numbers. This is completely characterized by the **cumulative distribution function (cdf)**. The cdf

$$F(t) := P(X \le t).$$

represents the probability that the random variable X is less than or equal to t. Then we say that "the random variable X is distributed as $F(t)$."

Properties of cdf. The cdf $F(t)$ must satisfy the following properties:

(a) The cdf $F(t)$ is a positive function.

$$F(t) \ge 0 \text{ for all } t.$$

(b) The cdf $F(t)$ increases monotonically.

$$F(s) \le F(t) \text{ whenever } s < t.$$

(c) The cdf $F(t)$ must tend to zero as t goes to negative infinity "$-\infty$", and must tend to one as t goes to positive infinity "$+\infty$."

$$\lim_{t \to -\infty} F(t) = 0$$
$$\lim_{t \to +\infty} F(t) = 1$$

Probability density function (pdf). It is often the case that the probability that the random variable X takes a value in a particular range is given by the area under a curve over that range of values. This curve is called the **probability density function (pdf)** of the random variable X, denoted by $f(x)$. Thus, the probability that "$a \le X \le b$" can be expressed by

$$P(a \le X \le b) = \int_a^b f(x)dx.$$

The pdf $f(x)$ must satisfy the following properties:

(a) The pdf $f(x)$ is a positive function [that is, $f(x) \ge 0$].

(b) The area under the curve of $f(x)$ (and above the x-axis) is one [that is, $\int_{-\infty}^{\infty} f(x)dx = 1$].

Relation between cdf and pdf. The pdf $f(x)$ is related to the cdf $F(t)$ via

$$F(t) = P(X \le t) = \int_{-\infty}^{t} f(x)dx.$$

This implies that such cdf $F(t)$ is a continuous function, and that the pdf $f(x)$ is the derivative of the cdf $F(x)$ if f is continuous at x, that is,

$$f(x) = \frac{dF(x)}{dx}.$$

Example 1. A random variable X is called a **uniform** random variable on $[a, b]$, when X takes any real number in the interval $[a, b]$ equally likely. Then the pdf of X is given by

$$f(x) = \begin{cases} 1/(b-a) & \text{if } a \le x \le b; \\ 0 & \text{otherwise [that is, if } x < a \text{ or } b < x]. \end{cases}$$

Find the cdf $F(t)$.

Solution.

$$F(t) = \int_{-\infty}^{t} f(x)dx = \begin{cases} 0 & \text{if } t < a; \\ \frac{t-a}{b-a} & \text{if } a \le t \le b; \\ 1 & \text{if } t > b. \end{cases}$$

Joint density function. Consider a pair (X, Y) of random variables. A **joint density function** $f(x, y)$ is a nonnegative function satisfying

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dy\, dx = 1,$$

and is used to compute probabilities constructed from the random variables X and Y simultaneously by

$$P(a \le X \le b, c \le Y \le d) = \int_{a}^{b} \left[\int_{c}^{d} f(x, y)\, dy \right] dx$$

Marginal densities and independence. Given the joint density function $f(x, y)$, the distribution for each of X and Y is called the **marginal distribution**, and the **marginal density functions** of X and Y, denoted by $f_X(x)$ and $f_Y(y)$, are given respectively by

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy \quad \text{and} \quad f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx.$$

If the joint density function $f(x, y)$ for continuous random variables X and Y is expressed in the form

$$f(x, y) = f_X(x) f_Y(y) \quad \text{for all } x, y,$$

then X and Y are said to be independent.

Expectations and Variances

Expectation. Let $f(x)$ be the pdf of a continuous random variable X. Then we define the expectation of the random variable X by

$$E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx.$$

Furthermore, we can define the expectation $E[g(X)]$ of the function $g(X)$ of random variable by

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)\, f(x)\, dx,$$

if $g(x)$ is integrable with respect to $f(x)\, dx$, that is, $\int_{-\infty}^{\infty} |g(x)|\, f(x)\, dx < \infty$.

Properties of expectation. When X is a discrete random variable, $E[X]$ is considered as a "linear operator." This remains true even if X is a continuous random variable. Thus, we have the following properties (without proof):

(a) $E[a + X] = a + E[X]$.

(b) $E[aX + bY] = aE[X] + bE[Y]$.

(c) $E\left[a + \sum_{i=1}^{n} b_i X_i\right] = a + \sum_{i=1}^{n} b_i E[X_i]$.

(d) If X and Y are independent then

$$E[g_1(X)g_2(Y)] = E[g_1(X)] \times E[g_2(Y)].$$

Variance. We can define the variance for the continuous random variable X by

$$\mathrm{Var}(X) := E[(X - E(X))^2] = E[X^2] - (E[X])^2,$$

and often denote it by σ^2. The square-root $\sqrt{\mathrm{Var}(X)}$ is the standard deviation (SD), denoted by σ.

Example 2. Let X be a uniform random variable on $[a, b]$. Compute $E[X]$ and $E[X^2]$, and find $\mathrm{Var}(X)$.

Solution.

$$\begin{aligned}
E[X] &= \int_a^b x \left(\frac{1}{b-a}\right) dx = \left(\frac{1}{b-a}\right)\left[\frac{x^2}{2}\right]_a^b \\
&= \frac{a+b}{2} \\
E[X^2] &= \int_a^b x^2 \left(\frac{1}{b-a}\right) dx = \left(\frac{1}{b-a}\right)\left[\frac{x^3}{3}\right]_a^b \\
&= \frac{a^2+ab+b^2}{3} \\
\mathrm{Var}(X) &= \frac{a^2+ab+b^2}{3} - \left(\frac{a+b}{2}\right)^2 = \frac{(b-a)^2}{12}
\end{aligned}$$

Exponential and Gamma Distributions

Exponential distribution. The exponential density function is defined as

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0; \\ 0 & x < 0 \end{cases}$$

Then the cdf is computed as

$$F(t) = \begin{cases} 1 - e^{-\lambda t} & t \geq 0; \\ 0 & t < 0 \end{cases}$$

Survival function and memoryless property. The function $S(t) = 1 - F(t) = P(X > t)$ is called the **survival function**. When $F(t)$ is an exponential distribution, we have $S(t) = e^{-\lambda t}$ for $t \geq 0$. Furthermore, we can find that

$$P(X > t + s \mid X > s) = \frac{P(X > t + s)}{P(X > s)} = \frac{S(t + s)}{S(s)} = S(t) = P(X > t), \tag{5.1}$$

which is referred as the **memoryless** property of exponential distribution.

Gamma distribution. The *gamma function*, denoted by $\Gamma(x)$, is defined as

$$\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u}\, du, \quad \alpha > 0.$$

It satisfies the recursive formula $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$. Then the *gamma density* is defined as

$$f(t) = \frac{\lambda^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\lambda t} \quad t \geq 0$$

which depends on two parameters $\alpha > 0$ and $\lambda > 0$.

Parameters of gamma distribution. We call the parameter α a **shape parameter**, because changing α changes the shape of the density. We call the parameter λ a **rate parameter**, because changing λ merely rescales the density without changing its shape. This is equivalent to changing the units of measurement (feet to meters, or seconds to minutes). Suppose that the parameter α is an integer n. Since $\Gamma(1) = 1$, by the recursive formula we obtain $\Gamma(n) = (n-1)!$. In particular, the gamma distribution with $\alpha = 1$ becomes an exponential distribution (with parameter λ).

Expectation of gamma distribution. Let X be a gamma random variable with parameter (α, λ). Then we can calculate $E[X]$ and $E[X^2]$ as follows.

$$\begin{aligned} E[X] &= \int_0^\infty x \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}\, dx = \frac{1}{\lambda\Gamma(\alpha)} \int_0^\infty (\lambda x)^\alpha e^{-\lambda x}\, \lambda dx \\ &= \frac{1}{\lambda\Gamma(\alpha)} \int_0^\infty u^\alpha e^{-u}\, du = \frac{1}{\lambda} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha)} = \frac{\alpha}{\lambda} \end{aligned}$$

Variance of gamma distribution. We have

$$E[X^2] = \int_0^\infty x^2 \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}\,dx = \frac{1}{\lambda^2\Gamma(\alpha)} \int_0^\infty (\lambda x)^{\alpha+1} e^{-\lambda x}\,\lambda dx$$
$$= \frac{1}{\lambda^2\Gamma(\alpha)} \int_0^\infty u^{\alpha+1} e^{-u}\,du = \frac{1}{\lambda^2}\frac{\Gamma(\alpha+2)}{\Gamma(\alpha)} = \frac{\alpha(\alpha+1)}{\lambda^2}$$

Thus, we can obtain

$$\mathrm{Var}(X) = \frac{\alpha(\alpha+1)}{\lambda^2} - \left(\frac{\alpha}{\lambda}\right)^2 = \frac{\alpha}{\lambda^2}$$

Chi-square distribution. The gamma distribution with $\alpha = \dfrac{n}{2}$ and $\lambda = \frac{1}{2}$ is called the **chi-square distribution** with n degrees of freedom. [It plays a vital role later in understanding another important distribution, called t-distribution.]

Normal Distributions

Normal distribution. A **normal distribution** is represented by a family of distributions which have the same general shape, sometimes described as "bell shaped." The normal distribution has the pdf

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right], \tag{5.2}$$

which depends upon two parameters μ and σ^2. In (5.2), $\pi = 3.14159\ldots$ is the famous "pi" (the ratio of the circumference of a circle to its diameter), and $\exp(u)$ is the exponential function e^u with the base $e = 2.71828\ldots$ of the natural logarithm.

Integrations of normal density functions. Much celebrated integrations are the following:

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = 1, \tag{5.3}$$
$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^\infty x e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = \mu, \tag{5.4}$$
$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^\infty (x-\mu)^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = \sigma^2. \tag{5.5}$$

Equation (5.3) guarantees that the function (5.2) always represents a "probability density" no matter what values the parameters μ and σ^2 would take. Equation (5.4) and (5.5) respectively imply $E[X] = \mu$ and $\mathrm{Var}(X) = \sigma^2$ for a normal random variable X.

Parameters of normal density function. We say that a random variable X is **normally distributed** with parameter (μ, σ^2) when X has the pdf (5.2). The parameter μ, called a **mean** (or, location parameter), provides the center of the density, and the density function $f(x)$ is symmetric around μ. The parameter σ is a **standard deviation** (or, a scale parameter); small values of σ lead to high peaks but sharp drops. Larger values of σ lead to flatter densities. The shorthand notation

$$X \sim N(\mu, \sigma^2)$$

is often used to express that X is a normal random variable with parameter (μ, σ^2).

Linear transform of normal random variable. One of the important properties of normal distribution is that if X is a normal random variable with parameter (μ, σ^2) [that is, the pdf of X is given by (5.2)], then $Y = aX + b$ is *also* a normal random variable having parameter $(a\mu + b, (a\sigma)^2)$. In particular,

$$\frac{X - \mu}{\sigma} \tag{5.6}$$

becomes a normal random variable with parameter $(0, 1)$, called the **standard normal distribution**.

Standard normal distribution. The normal density $\phi(x)$ with parameter $(0, 1)$ is given by

$$\phi(x) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},$$

and the **table of standard normal distribution** is used to obtain the values for the cdf

$$\Phi(t) := \int_{-\infty}^{t} \phi(x)\, dx.$$

How to calculate probabilities. Suppose that a tomato plant height X is normally distributed with parameter (μ, σ^2). Then what is the probability that the tomato plant height is between a and b? The integration

$$P(a \le X \le b) = \int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx$$

seems too complicated.

How to calculate probabilities, continued. If we consider the random variable $\dfrac{X - \mu}{\sigma}$, then the event $\{a \le X \le b\}$ is equivalent to the event

$$\left\{ \frac{a - \mu}{\sigma} \le \frac{X - \mu}{\sigma} \le \frac{b - \mu}{\sigma} \right\}.$$

Let $a' = \frac{a-\mu}{\sigma}$ and $b' = \frac{b-\mu}{\sigma}$. Then in terms of probability, this means that

$$P(a \le X \le b) = P\left(a' \le \frac{X - \mu}{\sigma} \le b' \right) = \int_{a'}^{b'} \phi(x)\, dx = \Phi(b') - \Phi(a'). \tag{5.7}$$

Finally look at the values for $\Phi(a')$ and $\Phi(b')$ from the table of standard normal distribution, and plug them into (5.7).

Summary. Suppose that a random variable X is normally distributed with mean μ and standard deviation σ. Then we can obtain

(a) $P(a \le X \le b) = \Phi\left(\dfrac{b - \mu}{\sigma}\right) - \Phi\left(\dfrac{a - \mu}{\sigma}\right);$

(b) $P(X \le b) = \Phi\left(\dfrac{b-\mu}{\sigma}\right)$;

(c) $P(a \le X) = 1 - \Phi\left(\dfrac{a-\mu}{\sigma}\right)$.

Example 3. The tomato plant height is normally distributed with parameters $\mu = 15$ and $\sigma = 4$ inches. What is the probability that the height is between 14.24 and 16.16 inches?

Solution. We can calculate $a' = \frac{14.24-15}{4} = -0.19$ and $b' = \frac{16.16-15}{4} = 0.29$. Then find $\Phi(-0.19) = 0.4247$ and $\Phi(0.29) = 0.6141$. Thus, the probability of interest becomes 0.1894, or approximately 0.19.

Central Limit Theorem

Convergence in distribution. Let $Z_1, Z_2, \ldots$ be a sequence of random variables having the cdf's $F_1, F_2, \ldots$, and let Z be a random variable having the cdf F. Then we say that the sequence X_n converges to X in distribution (in short, X_n converges to F) if

$$\lim_{n\to\infty} F_n(x) = F(x) \quad \text{for every } x \text{ at which } F(x) \text{ is continuous.}$$

The convergence is completely characterized in terms of the distributions $F_1, F_2, \ldots$ and F.

Central Limit Theorem. Let $X_1, X_2, \ldots$ be a sequence of "independent and identically distributed (iid)" random variables having the common distribution F with mean μ and variance σ^2. Then

$$Z_n = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sigma\sqrt{n}} = \frac{\sum_{i=1}^{n} (X_i - \mu)/\sigma}{\sqrt{n}} \qquad n = 1, 2, \ldots$$

converges to the standard normal distribution.

Normal approximation. Let $X_1, X_2, \ldots, X_n$ be "independent and identically distributed (iid)" random variables with mean μ and variance σ^2. If the size n is adequately large, then the distribution of the sum

$$Y = \sum_{i=1}^{n} X_i$$

can be approximated by the normal distribution with parameter $(n\mu, n\sigma^2)$. A general rule for "adequately large" n is about $n \ge 30$, but it is often good for much smaller n. Similarly the sample mean

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

has approximately the normal distribution with parameter $(\mu, \sigma^2/n)$.

Approximating probabilities. If n is sufficiently large, then the following normal approximation can be applied for the sum Y or the average $\bar{X}$. It allows us to approximate the probability that the random variable (either Y or $\bar{X}$) takes a value between a and b.

Variable	Approximation	Probability between a and b
$Y = X_1 + \cdots + X_n$	$N(n\mu, n\sigma^2)$	$\Phi\left(\frac{b - n\mu}{\sigma\sqrt{n}}\right) - \Phi\left(\frac{a - n\mu}{\sigma\sqrt{n}}\right)$
$\bar{X} = \frac{X_1 + \cdots + X_n}{n}$	$N\left(\mu, \frac{\sigma^2}{n}\right)$	$\Phi\left(\frac{b - \mu}{\sigma/\sqrt{n}}\right) - \Phi\left(\frac{a - \mu}{\sigma/\sqrt{n}}\right)$

Example 4. An insurance company has 10,000 automobile policyholders. The expected yearly claim per policyholder is \$240 with a standard deviation of \$800. Find approximately the probability that the total yearly claim exceeds \$2.7 million. Can you say that such event is highly unlikely?

Solution.

$$P\left(\sum_{i=1}^{10000} X_i > 2700000\right) \approx 1 - \Phi(3.75) < 0.0001$$

This is highly unlikely. **Approximation for binomial distributions.** Suppose that $X_1, \ldots, X_n$ are "independent and identically distributed (iid)" Bernoulli random variables with the mean $p = E(X)$ and the variance $p(1-p) = \mathrm{Var}(X)$. If the size n is adequately large, then the distribution of the sum

$$Y = \sum_{i=1}^{n} X_i$$

can be approximated by the normal distribution with parameter $(np, np(1-p))$. Thus, the normal distribution $N(np, np(1-p))$ approximates the binomial distribution $B(n,p)$. A general rule for "adequately large" n is to satisfy $np \geq 5$ and $n(1-p) \geq 5$.

calculating probabilities with correction. Let Y be a binomial random variable with parameter (n,p), and let X be a normal random variable with parameter $(np, np(1-p))$. Then the distribution of Y can be approximated by that of X. However, since X is a continuous random variable, the *continuity correction* has to be made in the following approximation of probability calculation.

$$\begin{aligned} P(i \leq Y \leq j) &\approx P(i - 0.5 \leq X \leq j + 0.5) \\ &= \Phi\left(\frac{j + 0.5 - np}{\sqrt{np(1-p)}}\right) - \Phi\left(\frac{i - 0.5 - np}{\sqrt{np(1-p)}}\right). \end{aligned}$$

Example 5. Suppose that a coin is tossed 100 times and lands heads 60 times or more. Should we be surprised and doubt that the coin is fair?

Solution.

$$\begin{aligned} P(60 \leq Y \leq 100) &\approx \Phi\left(\frac{100.5 - 50}{5}\right) - \Phi\left(\frac{59.5 - 50}{5}\right) \\ &= \Phi(9.9) - \Phi(1.9) = 1 - 0.9713 = 0.0287 \end{aligned}$$

It is highly unlikely, and we should doubt that the coin is fair.

Assignment No.5

Supplementary Readings.

SS: Murray R. Spiegel, John Schiller, and R. Alu Srinivasan, *Probability and Statistics 4th ed.* McGraw-Hill.

Chapter 2: Continuous Random Variables, Graphical Interpretations, Joint Distribution: Continuous Case,

Chapter 3: Definition of Mathematical Expectation, Functions of Random Variables.

Chapter 4: Normal Distribution, Some Properties of Normal Distribution, Relation between Binomial and Normal Distributions, Central Limit Theorem, Uniform Distribution, Gamma Distribution.

TH: Elliot A. Tanis and Robert V. Hogg, *A Brief Course in Mathematical Statistics*. Prentice Hall.

Section 3.2–3.4, and 3.6–3.7.

WM: Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, and Keying Ye, *Probability & Statistics for Engineers & Scientists*, 9th ed. Prentice Hall.

Section 3.1–3.4., 6.1–6.6, and 8.4

Problem 1. Suppose that X has the density function $f(x) = cx^2$ for $0 \le x \le 1$ and $f(x) = 0$ otherwise.

(a) Find c.

(b) Find the cdf.

(c) What is $P(.1 \le X < .5)$?

Problem 2. Let X be a random variable with the pdf

$$f(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1; \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $E[X]$.

(b) Find $E[X^2]$ and $\mathrm{Var}(X)$.

Problem 3. Suppose that the lifetime of an electronic component follows an exponential distribution with rate parameter $\lambda = 0.2$.

(a) Find the probability that the lifetime is less than 10.

(b) Find the probability that the lifetime is between 5 and 15.

Problem 4. Suppose that in a certain population, individual's heights are approximately normally distributed with parameters $\mu = 70$ and $\sigma = 3$in.

(a) What proportion of the population is over 6ft. tall?

(b) What is the distribution of heights if they are expressed in centimeters?

Problem 5. Let X be a normal random variable with $\mu = 5$ and $\sigma = 10$.

(a) Find $P(X > 10)$.

(b) Find $P(-20 < X < 15)$.

(c) Find the value of x such that $P(X > x) = 0.05$.

Problem 6. Suppose that $X \sim N(\mu, \sigma^2)$.

(a) Find $P(|X - \mu| \leq 0.675\sigma)$.

(b) Find the value of c in terms of σ such that $P(\mu - c \leq X \leq \mu + c) = 0.95$.

Problem 7. Consider a sample $X_1, \ldots, X_9$ of normally distributed random variables with mean μ and variance $\sigma^2 = 1$. What is the probability that $|\mu - \bar{X}| \leq 0.3$?

Problem 8. An actual voltage of new 1.5-volt battery has the probability density function

$$f(x) = 5, \quad 1.4 \leq x \leq 1.6.$$

Estimate the probability that the sum of the voltages from 120 new batteries lies between 170 and 190 volts.

Problem 9. The germination time in days of a newly planted seed has the probability density function

$$f(x) = 0.3e^{-0.3x}, \quad x \geq 0.$$

If the germination times of different seeds are independent of one another, estimate the probability that the average germination time of 2000 seeds is between 3.1 and 3.4 days.

Problem 10. Calculate the following probabilities by using normal approximation with continuity correction.

(a) Let X be a binomial random variable with $n = 10$ and $p = 0.7$. Find $P(X \geq 8)$.

(b) Let X be a binomial random variable with $n = 15$ and $p = 0.3$. Find $P(2 \leq X \leq 7)$.

(c) Let X be a binomial random variable with $n = 9$ and $p = 0.4$. Find $P(X \leq 4)$.

(d) Let X be a binomial random variable with $n = 14$ and $p = 0.6$. Find $P(8 \le X \le 11)$.

Answers

Problem 1.

(a) $\displaystyle\int_0^1 cx^2\,dx = \frac{c}{3} = 1$. Thus, we have $c = 3$.

(b) $F(t) = \begin{cases} 0 & \text{if } x < 0; \\ t^3 & \text{if } 0 \le x \le 1; \\ 1 & \text{if } x > 1. \end{cases}$

(c) $P(.1 \le X < .5) = \displaystyle\int_{0.1}^{0.5} 3x^2\,dx = \left[x^3\right]_{0.1}^{0.5} = 0.124$

Problem 2.

(a) $E[X] = \displaystyle\int_0^1 x(2x)\,dx = \left[\frac{2x^3}{3}\right]_0^1 = \frac{2}{3}$

(b) $E[X^2] = \displaystyle\int_0^1 x^2(2x)\,dx = \left[\frac{x^4}{2}\right]_0^1 = \frac{1}{2}$ Then we obtain $\mathrm{Var}(X) = E[X^2] - (E[X])^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18}$.

Problem 3. Let X be the life time of the electronic component.

(a) $P(X \le 10) = 1 - e^{-(0.2)(10)} \approx 0.8647$

(b) $P(5 \le X \le 15) = e^{-(0.2)(5)} - e^{-(0.2)(15)} \approx 0.3181$

Problem 4. Let X be a normal random variable with $\mu = 70$ and $\sigma = 3$.

(a) $P(X \ge (6)(12)) = 1 - \Phi\left(\dfrac{72-70}{3}\right) \approx 1 - \Phi(0.67) \approx 0.25$

(b) In centimeters, the height becomes $(2.54)X$, and it is normally distributed with $\mu = (2.54)(70) = 177.8$ and $\sigma = (2.54)(3) = 7.62$.

Problem 5.

(a) $P(X > 10) = 1 - \Phi\left(\dfrac{10-5}{10}\right) = 1 - \Phi(0.5) = 0.3085$.

(b) $P(-20 < X < 15) = \Phi\left(\dfrac{15-5}{10}\right) - \Phi\left(\dfrac{-20-5}{10}\right) = \Phi(1) - \Phi(-2.5) = 0.8351.$

(c) Since $P(X > x) = 1 - \Phi\left(\dfrac{x-5}{10}\right) = 0.05$, we must have $\Phi\left(\frac{x-5}{10}\right) = 0.95$. By using the normal distribution table, we can find $\dfrac{x-5}{10} \approx 1.64$. Thus, $x \approx 5 + (10)(1.64) = 21.4$.

Problem 6.

(a) $P(|X-\mu| \le 0.675\sigma) = P(-0.675\sigma \le X - \mu \le 0.675\sigma) = \Phi(0.675) - \Phi(-0.675) \approx 0.5$

(b) $P(\mu - c \le X \le \mu + c) = 0.95$ implies $P\left(\dfrac{X-\mu}{\sigma} \le \dfrac{c}{\sigma}\right) = \Phi\left(\dfrac{c}{\sigma}\right) = 0.975$. Then we can find $\dfrac{c}{\sigma} \approx 1.96$. Thus, $c = 1.96\sigma$.

Problem 7. By the central limit theorem, $\bar{X}$ has a normal distribution with mean μ and variance $\frac{1}{9}$.

$$\begin{aligned} P(|\mu - \bar{X}| \le 0.3) &= P(-0.3 + \mu \le \bar{X} \le 0.3 + \mu) \\ &= \Phi\left(\frac{0.3}{1/3}\right) - \Phi\left(\frac{-0.3}{1/3}\right) \\ &= \Phi(0.9) - \Phi(-0.9) = 0.6318 \end{aligned}$$

Problem 8. Let $X_1, \ldots, X_{120}$ be the voltage of each battery, having the pdf $f(x)$. Since $\mu = E[X_i] = \frac{1.4+1.6}{2} = 1.5$ and $\sigma^2 = \frac{(1.6-1.4)^2}{12} = 0.0033$, the sum $Y = \sum_{i=1}^{120} X_i$ is approximately distributed as $N(120\mu, 120\sigma^2) = N(180, 0.4)$. Thus, we obtain

$$P(170 \le Y \le 190) = \Phi\left(\frac{190-180}{\sqrt{0.4}}\right) - \Phi\left(\frac{170-180}{\sqrt{0.4}}\right) \approx 1$$

Problem 9. Let $X_1, \ldots, X_{2000}$ be the germination time of individual seed, having the exponential distribution with $\lambda = 0.3$. Since $\mu = E[X_i] = 1/0.3 \approx 3.33$ and $\sigma^2 = \mathrm{Var}(X_i) = 1/(0.3)^2 \approx 11.11$, the sample mean $\bar{X}$ is approximately distributed as $N(\mu, \sigma^2/n) = N(3.33, 0.0056)$. Thus,

$$P(3.1 \le \bar{X} \le 3.4) = \Phi\left(\frac{3.4-3.33}{\sqrt{0.0056}}\right) - \Phi\left(\frac{3.1-3.33}{\sqrt{0.0056}}\right) \approx 0.82$$

Problem 10.

(a) $P(X \ge 8) = 1 - \Phi\left(\dfrac{7.5-(10)(0.7)}{\sqrt{(10)(0.7)(0.3)}}\right) \approx 0.365$

(b) $P(2 \le X \le 7) = \Phi\left(\frac{7.5-(15)(0.3)}{\sqrt{(15)(0.3)(0.7)}}\right) - \Phi\left(\frac{1.5-(15)(0.3)}{\sqrt{(15)(0.3)(0.7)}}\right) \approx 0.909$

(c) $P(X \le 4) = \Phi\left(\frac{4.5-(9)(0.4)}{\sqrt{(9)(0.4)(0.6)}}\right) \approx 0.7299$

(d) $P(8 \le X \le 11) = \Phi\left(\frac{11.5-(14)(0.6)}{\sqrt{(14)(0.6)(0.4)}}\right) - \Phi\left(\frac{7.5-(14)(0.6)}{\sqrt{(14)(0.6)(0.4)}}\right) \approx 0.6429$