Sample Space and Events

Sample space.

Some textbook uses S instead

Upper case 'omega'

The set of all possible outcomes of an experiment is called the **sample space**, and is typically denoted by Ω. For example, if the outcome of an experiment is the order of finish in a race among 3 boys, Jim, Mike and Tom, then the sample space becomes

$$\Omega = \{(J, M, T), (J, T, M), (M, J, T), (M, T, J), (T, J, M), (T, M, J)\}.$$

outcome ⟷ Jim wins 1st place, Tom wins 2nd

In other example, suppose that a researcher is interested in the lifetime of a transistor, and measures it in minutes. Then the sample space is represented by

$$\Omega = \{\text{all nonnegative real numbers}\}.$$

20.3 or 410.5

Events.

Any subset of the sample space is called an **event**. In the who-wins-the-race example, "Mike wins the race" is an event, which we denote by A. Then we can write

$$A = \{(M, J, T), (M, T, J)\}.$$

Simply a subset of Ω

In the transistor example, "the transistor does not last longer than 2500 minutes" is an event, which we denote by E. And we can write

$$E = \{x : 0 \leq x \leq 2500\}.$$

It is a subset of Ω.

Set operations.

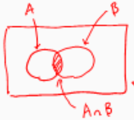
Once we have events $A, B, \ldots$, we can define a new event from these events by using set operations—union, intersection, and complement. The event $A \cup B$, called the **union** of A and B, means that either A or B or both occurs. Consider the "who-wins-the-race" example, and let B denote the event "Jim wins the second place", that is,

$$B = \{(M, J, T), (T, J, M)\}.$$

Then $A \cup B$ means that "either Mike wins the first, or Jim wins the second, or both," that is,

$$A \cup B = \{(M, T, J), (T, J, M), (M, J, T)\}.$$

The event $A \cap B$, called the **intersection** of A and B, means that both A and B occurs. In our example, $A \cap B$ means that "Mike wins the first and Jim wins the second," that is,



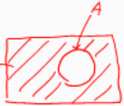
$$A \cap B = \{(M, J, T)\}.$$

Set operations, continued.

'A complement' Some uses the notation $\bar{A}$

The event A^c, called the **complement** of A, means that A does not occur. In our example, the event A^c means that "Mike does not win the race", that is,

$$A^c = \{(J, M, T), (J, T, M), (T, J, M), (T, M, J)\}.$$



Now suppose that C is the event "Tom wins the race." Then what is the event $A \cap C$? It is impossible that Mike wins and Tom wins at the same time. In mathematics, it is called the **empty set**, denoted by $\emptyset$, and can be expressed in the form

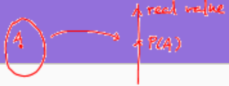
$$A \cap C = \emptyset.$$

or ϕ

If the two events A and B satisfy $A \cap B = \emptyset$, then they are said to be **disjoint**. For example, "Mike wins the race" and "Tom wins the race" are disjoint events.

Axioms of Probability

Axioms of probability.

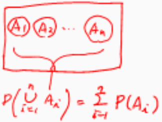


A probability P is a "function" defined over all the "events" (that is, all the subsets of Ω), and must satisfy the following properties:

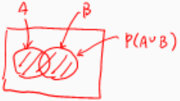
1. If $A \subset \Omega$, then $0 \leq P(A) \leq 1$
2. $P(\Omega) = 1$
3. If $A_1, A_2, \ldots$ are events and $A_i \cap A_j = \emptyset$ for all $i \neq j$ then, $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$.

Property (c) is called the "axiom of countable additivity," which is clearly motivated by the property:

$$P(A_1 \cup \cdots \cup A_n) = P(A_1) + \cdots + P(A_n)$$

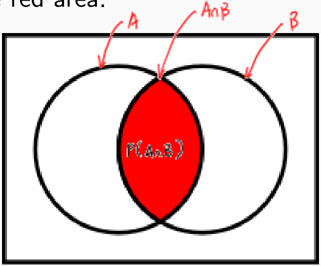


if $A_1, A_2, \ldots, A_n$ are mutually exclusive events. However, there is nothing in one's intuitive notion that requires this axiom.

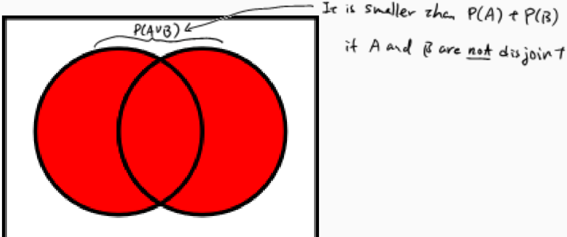


Note that if A and B are not disjoint then $P(A \cup B) < P(A) + P(B)$

The following figure, called **Venn diagram**, represents the probability of $A \cap B$ as shown in the red area.



The next figure now indicates the probability of $A \cup B$ in the red area.

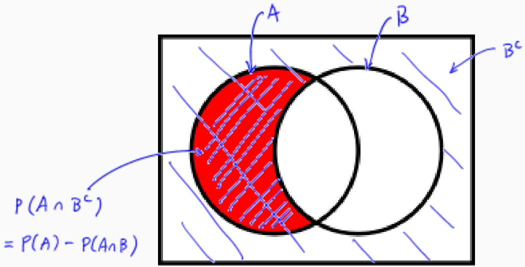


Rules of probability.

A visual illustration of Venn diagram allows us to devise the following addition rule which is not obvious from the axioms of probability.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Another rule $P(A \cap B^c) = P(A) - P(A \cap B)$ can be observed from the following figure.



Sample space having equally likely outcomes.

When the sample space

lower case 'omega'

$$\Omega = \{\omega_1, \ldots, \omega_N\}$$

$A_i = \{\omega_i\}$

consists of N outcomes, it is often natural to assume that $P(\{\omega_1\}) = \cdots = P(\{\omega_N\})$. Then we can obtain the probability of a single outcome by

$P(\Omega) = 1$

$$P(\{\omega_i\}) = \frac{1}{N} \quad \text{for } i = 1, \ldots, N.$$

Assuming this, we can compute the probability of any event A by counting the number of outcomes in A, and obtain

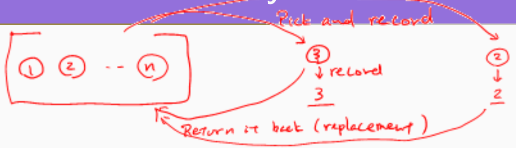
$$P(A) = \frac{\text{number of outcomes in } A}{N}.$$

Example

$A = \{w_1, w_2, w_3\}$

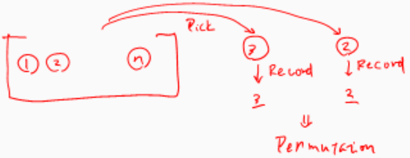
$P(A) = P(\{w_1\} \cup \{w_2\} \cup \{w_3\}) = P(\{w_1\}) + P(\{w_2\}) + P(\{w_3\}) = \frac{3}{N}$

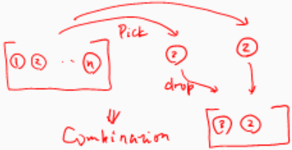
Combinatorial analysis.

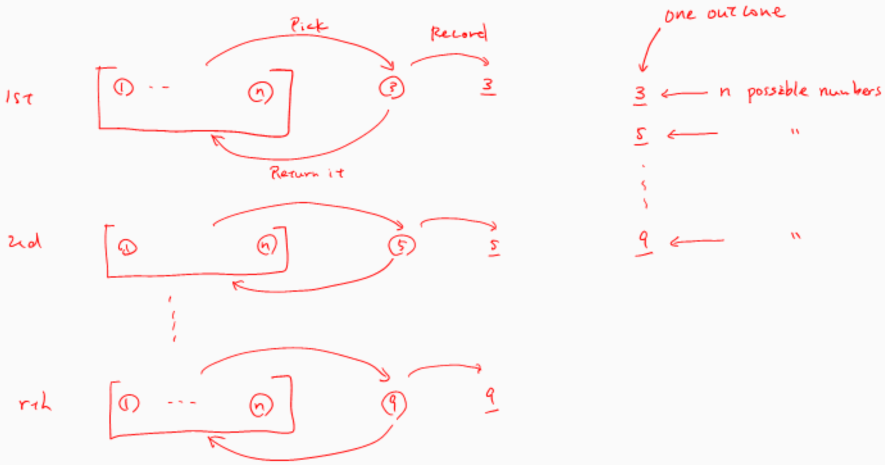


A problem involving equally likely outcomes can be solved by counting. The mathematics of counting is known as **combinatorial analysis.** It can be summarized in three basic methodologies:

1. *Multiplication principle* when individual objects are sampled with replacement and ordered.
2. *Permutations* when individual objects are sampled *without* replacement and ordered.
3. *Combinations* when a subset of individual objects (unordered) are sampled *without* replacement.






Pick
Record
Return it
1st
2nd
rth
one outcome
n possible numbers

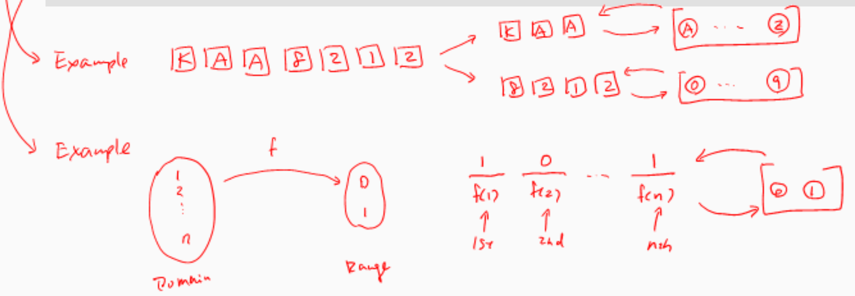
Multiplication principle.

If one experiment has m outcomes, and a second has n outcomes, then there are $m \times n$ outcomes for the two experiments. The **multiplication principle** can be extended and used in the following experiment. Suppose that we have n differently numbered balls in an urn. In the first trial, we pick up a ball from the urn, record its number, and put it back into the urn. In the second trial, we again pick up, record, and put back a ball. Continue the same procedure until the r-th trial. The whole process is called a **sampling with replacement**. By generalizing the multiplication principle, the number of all possible outcomes becomes

$$\underbrace{n \times \cdots \times n}_{r} = n^r.$$

Example

1. How many different 7-place license plates are possible if the first 3 places are to be occupied by letters and the final 4 by numbers?
2. How many different functions f defined on $\{1, \ldots, n\}$ are possible if the value $f(i)$ takes either 0 or 1?



Example

1. How many different 7-place license plates are possible if the first 3 places are to be occupied by letters and the final 4 by numbers?
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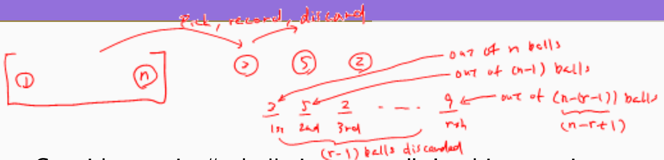
1. $\underbrace{26 \times 26 \times 26}_{\text{3 letters}} \times \underbrace{10 \times 10 \times 10 \times 10}_{\text{4 numbers}} = 175,760,000$
2. n places have either 0 or 1. Thus, we have $\underbrace{2 \times 2 \times \cdots \times 2}_{n} = 2^n$

Combinations and Binomial Coefficients

They are equal

Permutations.

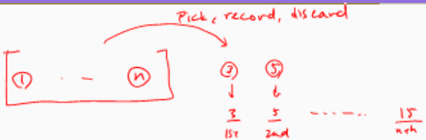


Consider again "n balls in an urn." In this experiment, we pick up a ball from the urn, record its number, but do not put it back into the urn. In the second trial, we again pick up, record, and do not put back a ball. Continue the same procedure r-times. The whole process is called a **sampling without replacement**. Then, the number of all possible outcomes becomes

$$\underbrace{n \times (n-1) \times (n-2) \times \cdots \times (n-r+1)}_{r}$$

$$= \frac{n \times (n-1) \times \cdots \times (n-r+1) \times (n-r) \times \cdots \times 3 \times 2 \times 1}{(n-r) \times (n-r-1) \times \cdots \times 3 \times 2 \times 1} = \frac{n!}{(n-r)!}$$

Permutations, continued.



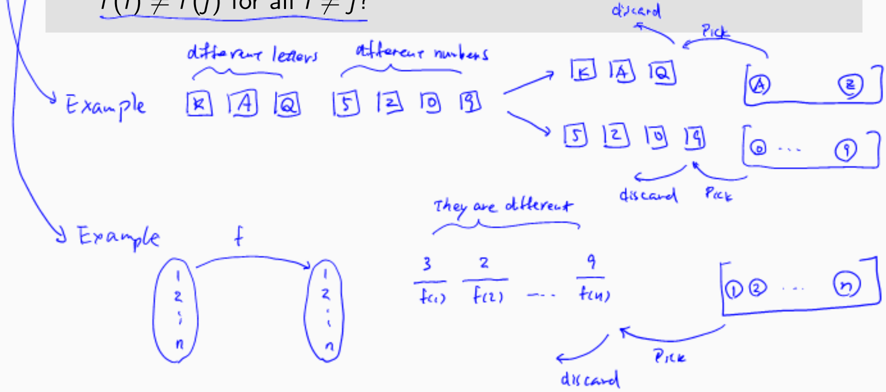
Given a set of n different elements, the number of all the possible "ordered" sets of size r is called r-**element permutation** of an n-element set. In particular, the n-element permutation of an n-element set is expressed as

$$\underbrace{n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1}_{n} = n!$$

The above mathematical symbol "n!" is called "n factorial." We define $0! = 1$, since there is one way to order 0 elements.

Example

1. How many different 7-place license plates are possible if the first 3 places are using different letters and the final 4 using different numbers?
2. How many different functions f defined on $\{1, \ldots, n\}$ are possible if the function f takes values on $\{1, \ldots, n\}$ and satisfies that $f(i) \neq f(j)$ for all $i \neq j$?



Example

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1. Here different places must have different objects. Thus, we have

$$\underbrace{26 \times 25 \times 24}_{\text{3 different letters}} \times \underbrace{10 \times 9 \times 8 \times 7}_{\text{4 different numbers}} = 78,624,000$$

2. n places must have different numbers from 1 to n. Thus, we have

$$\underbrace{n \times (n-1) \times \cdots \times 2 \times 1}_{\text{n different numbers}} = n!$$

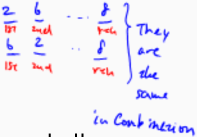
Combinations.

In the experiment, we have n differently numbered balls in an urn, and pick up r balls "as a group." Then there are

Permutation

$$\frac{n!}{(n-r)!}$$

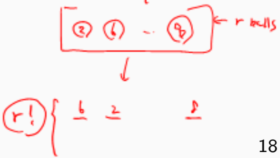
outcomes are different



ways of selecting the group if the order is relevant (r-element permutation). However, the order is irrelevant when you choose r balls as a group, and any particular r-element group has been counted exactly $r!$ times. Thus, the number of r-element groups can be calculated as

$$_nC_r = \binom{n}{r} = \frac{n!}{(n-r)!r!}.$$

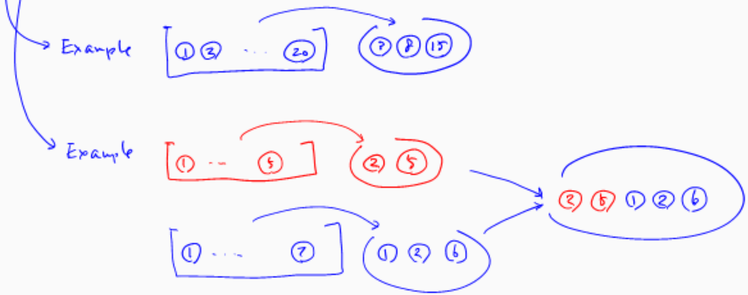
How many times does it repeat?



You should read the above symbol as "n choose r."

Example

1. A committee of 3 is to be formed from a group of 20 people. How many different committees are possible?
2. From a group of 5 women and 7 men, how many different committees consisting of 2 women and 3 men can be formed?



Example

1. A committee of 3 is to be formed from a group of 20 people. How many different committees are possible?
2. From a group of 5 women and 7 men, how many different committees consisting of 2 women and 3 men can be formed?

Permutation

1. $\binom{20}{3} = \frac{20 \times 19 \times 18}{3 \times 2 \times 1} = 1140$ $= r!$
2. Here you need to apply the multiplication principle together with combinations.

$$\underbrace{\binom{5}{2}}_{\text{2 women}} \times \underbrace{\binom{7}{3}}_{\text{3 men}} = 350$$

Binomial theorem.

The term "n choose r" is often referred as a binomial coefficient, because of the following identity.

combination

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i\, b^{n-i}.$$

In fact, we can give a proof of the binomial theorem by using combinatorial analysis. For a while we pretend that "commutative law" cannot be used for multiplication. Then, for example, the expansion of $(a+b)^2$ becomes

$\Vert\ (a+b)(a+b) ==$

$$aa + ab + ba + bb, \quad = a^2 + 2ab + b^2$$

binomial coefficient $\binom{2}{1}$

and consists of the 4 terms, aa, ab, ba, and bb. In general, how many terms in the expansion of $(a+b)^n$ should contain a's exactly in i places? The answer to this question indicates the binomial theorem.

Why does combination express binomial coefficient?

$$(a+b)^3 = (a+b)(a+b)(a+b)$$

$$= aaa + aab + aba + baa + abb + bab + bba + bbb$$

$aab + aba + baa$: $\binom{3}{2} a^2 b$

$abb + bab + bba$: $\binom{3}{1} a b^2$

$\underline{a}$ $\underline{a}$ $\underline{b}$
1st 2nd 3rd

↓

Choose 1st and 2nd place for a

↓

How many different ways?

$$\binom{3}{2} = \frac{3 \times 2}{2 \times 1} = 3$$

$\underline{a}$ $\underline{b}$ $\underline{b}$
1st 2nd 3rd

↓

How many different ways?

$$\binom{3}{1} = \frac{3}{1} = 3$$

Assignment No.1

Supplementary Readings.

SS: Murray R. Spiegel, John Schiller, and R. Alu Srinivasan, *Probability and Statistics 4th ed.* McGraw-Hill.
Chapter 1: Random Experiments, Sample Spaces, Events, Concept of Probability, Axioms of Probability, Some Important Theorems on Probability, Assignment of Probabilities, Combinatorial Analysis, Fundamental Principle of Counting, Permutations, Combinations, Binomial Coefficient.

TH: Elliot A. Tanis and Robert V. Hogg, *A Brief Course in Mathematical Statistics*. Prentice Hall.
Section 1.1–1.2

WM: Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, and Keying Ye, *Probability & Statistics for Engineers & Scientists*, 9th ed. Prentice Hall.
Section 2.1–2.5

Problem

The world series in baseball continues until either the american league (A) or the national league (N) wins four games. How many different outcomes are possible if the series goes

1. *four games?*
2. *five games?*
3. *six games?*
4. *seven games?*

For example, ANNAAA means that the american league wins in six games.

Problem

Two six-sided dice are thrown sequentially, and the face values that come up are recorded.

1. *List the sample space* Ω.
2. *List the elements that make up the following events:*
   2.1 $A = \{\textit{the sum of the two values is at least } 8\}$,
   2.2 $B = \{\textit{the value of the first die is higher than the value of the second}\}$,
   2.3 $C = \{\textit{the first value is } 4\}$.
3. *Assuming equally likely outcomes, find* $P(A)$, $P(B)$, *and* $P(C)$.
4. *List the events of the following events:*
   4.1 $A \cap C$,
   4.2 $B \cup C$,
   4.3 $A \cap (B \cup C)$.
5. *Again assuming equally likely outcomes, find* $P(A \cap C)$, $P(B \cup C)$, *and* $P(A \cap (B \cup C))$.

Problem

Suppose that $P(A) = 0.4$, $P(B) = 0.5$, *and* $P(A \cap B) = 0.3$. *Then find the following probability:*

1. $P(A \cup B)$
2. $P(A \cap B^c)$
3. $P(A^c \cup B^c)$

Problem

A deck of 52 *cards is shuffled thoroughly. What is the probability that the four aces are all next to each other? (Hint: Imagine that you have* 52 *slots lined up to place the four aces. How many different ways can you "choose" four slots for those aces? How many different ways do you get consecutive slots for those aces?)*

Problem

Express the answer by combinations in each of the following questions:

1. *How many ways are there to encode the 26-letter English alphabet into 8-bit binary words (sequences of eight 0's and 1's)?*
2. *What is the coefficient of* x^3y^4 *in the expansion of* $(x+y)^7$ *?*
3. *A child has six blocks, three of which are red and three of which are green. How many patterns can she make by placing them all in a line?*

Problem

From a group of 5 students, Amanda, Brad, Carey, David and Eric, we want to form a committee consisting of 3 students.

1. *How many different ways to choose committee members?*
2. *Now suppose that Amanda and Brad refuse to serve together. Then how many different ways to choose committee members?*

Answers

Problem 1.

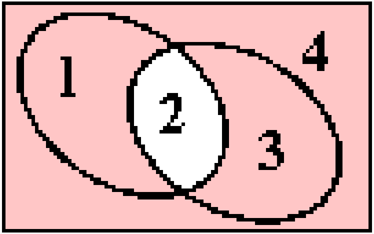
1. The series goes only four games only when AAAA or NNNN. Thus, only two outcomes.
2. The series goes five games if $\square\square\square\square A$ with only one N in $\square$'s, or $\square\square\square\square N$ with only one A in $\square$'s. Thus, we have $4 + 4 = 8$ outcomes.
3. The series goes six games if $\square\square\square\square\square A$ with exactly two N's in $\square$'s, or $\square\square\square\square\square N$ with exactly two A's in $\square$'s. Thus, we have $\binom{5}{2} + \binom{5}{2} = 20$ outcomes.
4. By now you must get the idea. The answer is $\binom{6}{3} + \binom{6}{3} = 40$ outcomes.

Problem 2.

1. $\Omega = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),$
   $(2,2),\dots,(6,6)\}$.
   Note that $(1,2)$ and $(2,1)$ are distinct outcomes, and that the part "$\dots$" should be obvious to you.
2. 2.1 $A = \{(2,6),(3,5),(3,6),(4,4),(4,5),(4,6),(5,3),\cdots,(5,6),$
   $(6,2),\dots,(6,6)\}$
   2.2 $B = \{(2,1),(3,1),(3,2),(4,1),\dots,(4,3),(5,1),\dots,(5,4),$
   $(6,1),\dots,(6,5)\}$
   2.3 $C = \{(4,1),\dots,(4,6)\}$
3. $P(A) = \frac{15}{36} = \frac{5}{12}$, $P(B) = \frac{15}{36} = \frac{5}{12}$, and $P(C) = \frac{6}{36} = \frac{1}{6}$.
4. 4.1 $A \cap C = \{(4,4),(4,5),(4,6)\}$
   4.2 $B \cup C = \{(2,1),(3,1),(3,2),(4,1),\dots,(4,6),(5,1),\dots,(5,4),$
   $(6,1),\dots,(6,6)\}$
   4.3 $A \cap (B \cup C) = \{(4,4),(4,5),(4,6),(5,3),(5,4),(6,2),\dots,(6,5)\}$
5. $P(A \cap C) = \frac{3}{36} = \frac{1}{12}$, $P(B \cup C) = \frac{18}{36} = \frac{1}{2}$, and
   $P(A \cap (B \cup C)) = \frac{9}{36} = \frac{1}{4}$.

Problem 3.

1. $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.6$
2. $P(A \cap B^c) = P(A) - P(A \cap B) = 0.1$
3. Draw the Venn diagram for the probability of $A^c \cup B^c$, and find the following rule.



$$P(A^c \cup B^c) = 1 - P(A \cap B) = 0.7$$

Problem 4.

Let Ω be the sample space of different outcomes in which four places are chosen for aces out of 52 places, and let A be the event that four places are located next to each other. Then the number of outcomes in Ω is $\binom{52}{4} = 270,725$, and the number of outcomes in A is just 49. Thus, the probability is calculated as $\frac{49}{270725} \approx 0.0002$ (which is very small).

Problem 5.

1. $\binom{2^8}{26} = \binom{256}{26}$
2. $\binom{7}{3}$
3. $\binom{6}{3}$

Problem 6.

1. $\binom{5}{3} = 10$
2. There are three possible outcomes in which Amanda and Brad serve together. Thus, by removing these three cases, we have $10 - 3 = 7$ different ways to form a committee.